Continuous distributions.

$$P(a<X<b)=\int_a^b f(x)\,dx$$

$$X\sim N(\mu,\sigma) \iff Z=\frac{X-\mu}{\sigma}\sim N(0,1)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \longleftrightarrow f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Ex. 8. X - household income

$$X\sim N(\mu,\sigma)$$

$$\mu=900$$

$$\sigma=200$$

$$P(X\in(600,1200))=P(600<X<1200)=\int_{600}^{1200}\frac{1}{200\sqrt{2\pi}}\cdot e^{-\frac{(x-900)^2}{2\cdot 200^2}}\,dx=F(1200)-F(600)=0.866$$

pnorm(1200, 900, 200) - pnorm(600, 900, 200) → in R

$$P(X<q)=0.03 \iff q=523.84 \qquad \int_{-\infty}^{q}\frac{1}{200\sqrt{2\pi}}\cdot e^{-\frac{(x-900)^2}{2\cdot 200^2}}\,dx=0.03$$

$X_1, X_2, \dots, X_n$ – independent r.v.s. with the same distribution

$$E[X_i]=\mu, \quad SD[X_i]=\sigma \quad (V[X_i]=\sigma^2)$$

$$\bar{X} = \frac{X_1+X_2+\dots+X_n}{n}, \quad E[\bar{X}] = \frac{1}{n}E[X_1+X_2+\dots+X_n] = \frac{1}{n}\left(E[X_1]+E[X_2]+\dots+E[X_n]\right) =$$

$$= \frac{1}{n}(\mu+\mu+\dots+\mu) = \frac{1}{n}\cdot n\mu = \mu$$

$$V[\bar{X}] = \frac{1}{n^2}\underbrace{V[X_1+\dots+X_n]}_{} \underset{\text{indep.}}{\overset{X_1,\dots,X_n}{=}} \frac{1}{n^2}\left(V[X_1]+V[X_2]+\dots+V[X_n]\right) = \frac{1}{n^2}\left(\sigma^2+\sigma^2+\dots+\sigma^2\right) =$$

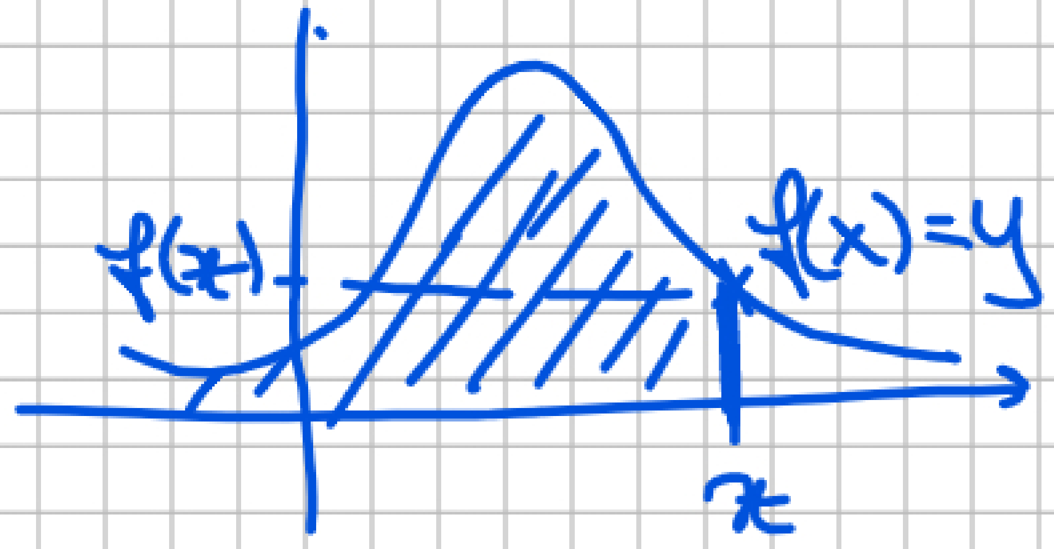
$$= \frac{1}{n^2}\cdot \underbrace{n\sigma^2}_{} = \frac{\sigma^2}{n} \iff SD[\bar{X}] = \frac{\sigma}{\sqrt{n}}$$

Ex. 10 X_i - the size of image i, $i=\overline{1,300}$

- indep.

$E[X_i]=1$, $SD[X_i]=0.5$

$$\bar{X} = \frac{X_1+X_2+\dots+X_{300}}{300} \sim N\left(\mu, \frac{\sigma}{\sqrt{300}}\right) \quad \mu=1, \sigma=0.5 \quad \iff \quad X_1+\dots+X_{300} \sim N(300\mu, \sqrt{300}\,\sigma)$$



$X_1+X_2+\dots+X_{300}$ - size of the 300 images

$$P(X_1+X_2+\dots+X_{300} \le 330) = P\left(\frac{X_1+X_2+\dots+X_{300}}{300} \le \frac{330}{300}\right) = P\left(\bar{X} \le \overset{\text{"}1.1}{\frac{33}{30}}\right) = \int_{-\infty}^{1.1} \frac{1}{\frac{0.5}{\sqrt{300}}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(x-1)^2\cdot 300}{2\cdot\sigma^2}} dx$$

$$= F(1.1) = 0.99$$

dnorm$(x,\mu,\sigma) \to f(x)$ (PDF)

pnorm$(q,\mu,\sigma) \to F(x)$ CDF

qnorm$(p,\mu,\sigma) \to F^{-1}(x)$

rnorm$(n,\mu,\sigma) \to$ generates random values

Ex. 11 X_i - weight of passenger i, $i=\overline{1,10}$

$P(X_1+X_2+\dots+X_{10}+150 \le 2000) = P(X_1+\dots+X_{10} \le 1850) = P(\bar{X} \le 185) = F(185)$

$\Downarrow$

pnorm$(185, 165, \frac{20}{\sqrt{10}})$

$X_i \sim N(\mu, \sigma)$ $\mu = 165$, $\sigma = 20$

$X_1+X_2+\dots+X_{10} \sim N(n\mu, \sqrt{n}\cdot\sigma)$, $\frac{X_1+X_2+\dots+X_{10}}{10} \sim N(\mu, \frac{\sigma}{\sqrt{10}})$

Ex. 12. X - nr. of damaged files

(a,b)

$\downarrow$

$(a-0.5, b+0.5)$

$X \sim Bin(n,p)$ $n=200$, $p=0.2$

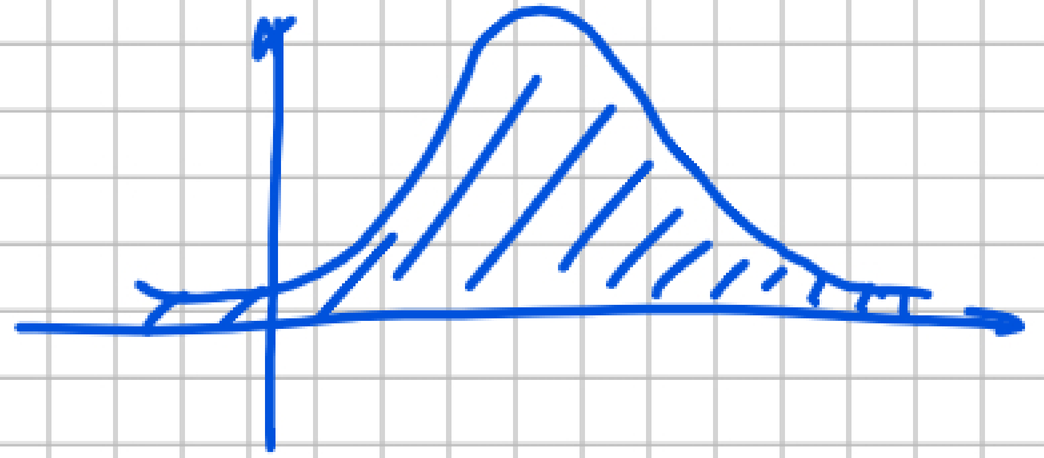
$P(X<50) = \sum_{k=0}^{49} \binom{n}{k} p^k (1-p)^{n-k} = F(49) = 0.95$ //

$Bin(n,p) \sim N(np, \sqrt{np(1-p)})$

$P(X<50) = P(X<50+0.5) = P(X<50.5) = \int_{-\infty}^{50.5} f(x)dx = 0.97$ //

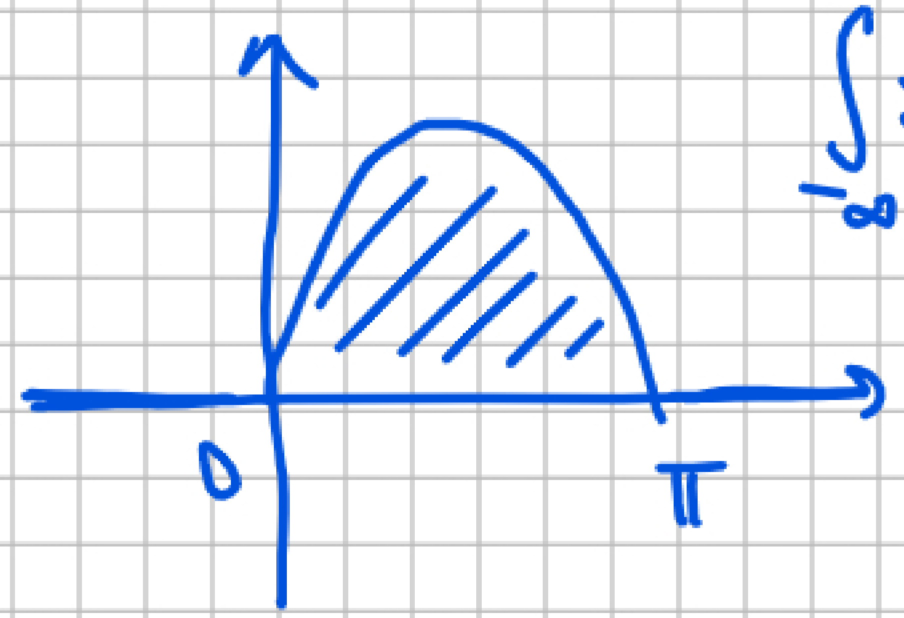
Consider the function:

$$f(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ a\sin x & , \text{ if } x \in [0, \pi] \\ 0 & , \text{ if } x > \pi \end{cases}$$



$P(-\infty < X < \infty) = 1$

$\int_{-\infty}^{\infty} f(x)dx = 1$



a) Find the coefficient a such that f is the density function of a continuous random variable X, and compute the distribution function of X.
b) Compute $E(X)$ and $V(X)$.
c) What is the probability that X takes values inside the interval $(0, \frac{\pi}{4})$?

a) $\int_{-\infty}^{\infty} f(x)dx = 1 \iff \int_0^{\pi} a\sin(x)dx = 1 \iff a(-\cos(x))\Big|_0^{\pi} = 1 \iff -a(-1-1) = 1 \iff 2a = 1 \iff$

$a = \frac{1}{2}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & , x < 0 \\ \frac{1}{2}(1-\cos x) & , x \in [0, \pi] \\ 1 & , x > \pi \end{cases}$$

$\int_{-\infty}^{x} f(t)\,dt = \int_0^{x} \frac{1}{2}\sin(t)dt = -\frac{1}{2}\cos(t)\Big|_0^{x} = -\frac{1}{2}(\cos x - 1) = \frac{1}{2}(1-\cos x)$

$\int_{-\infty}^{x} f(t)\,dt = \int_0^{\pi} f(t)\,dt = 1 \quad \left(\int_0^{\pi} \frac{1}{2}\sin t\,dt = -\frac{1}{2}\cos(t)\Big|_0^{\pi} = -\frac{1}{2}(-1-1) = 1\right)$

b) $E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{2}\int_0^{\pi} x \cdot \sin(x)\,dx = -\frac{1}{2} x\cos x \Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos x\,dx =$

$f(x) = x \quad \Rightarrow \quad f'(x) = 1$

$g'(x) = \sin(x) \qquad g(x) = -\cos x$

$= -\frac{1}{2}(-\pi) + \frac{1}{2}\sin x \Big|_0^{\pi} = \frac{1}{2}\pi$

$V[x] = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E^2[x] = \int_0^{\pi} x^2 \cdot \frac{1}{2}\sin(x)\,dx - \frac{\pi^2}{4} = \dots$

c) $P\left(x \in \left(0, \frac{\pi}{4}\right)\right) = \int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(x)\,dx = -\frac{1}{2}\cos x \Big|_0^{\frac{\pi}{4}} = -\frac{1}{2}\left(\frac{\sqrt{2}}{2} - 1\right) = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{2-\sqrt{2}}{4}$