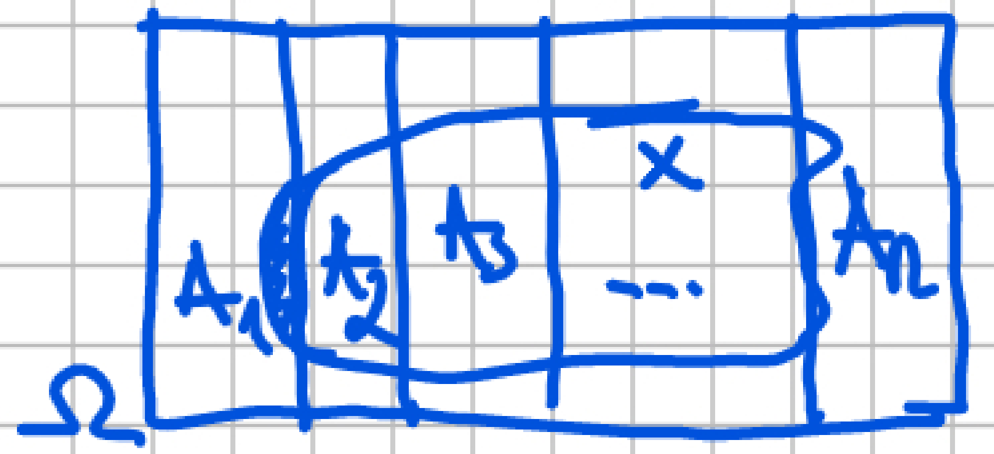
Lecture 2. Random variables. Discrete distributions.

Law of total probability

$A_1, A_2, \dots, A_n$ - a partition of Ω : $\begin{cases} A_i \cap A_j = \emptyset, \ (\forall)\, i \neq j \\ A_1 \cup A_2 \cup \dots \cup A_n = \Omega \end{cases}$



$X \subseteq \Omega$ - event

$$\underline{P(X)} = P(X \cap \Omega) = P(X \cap (A_1 \cup A_2 \cup \dots \cup A_n)) = P((X\cap A_1) \cup (X \cap A_2) \cup \dots \cup (X \cap A_n)) =$$

$X = X \cap \Omega$

$$= P(X \cap A_1) + P(X \cap A_2) + \dots + P(X \cap A_n) = \underline{P(X|A_1) \cdot P(A_1)} + P(X|A_2) \cdot P(A_2) + \dots + P(X|A_n) \cdot P(A_n)$$

$$P(X|A_1) = \frac{P(X \cap A_1)}{P(A_1)} \Leftrightarrow \underline{P(X \cap A_1)} = P(X|A_1) \cdot P(A_1)$$

$\Rightarrow P(X) = P(X|A_1)\cdot P(A_1) + P(X|A_2)\cdot P(A_2) + \dots + P(X|A_n)\cdot P(A_n)$

(law of total probability)

$P(X|A_i)$ – "probability of X given A_i"

– probability that X occurs knowing that A_i already occured

Ex 1. 3 boxes : $\begin{cases} \text{2 white balls, 3 black balls} \to \text{box 1} \\ \text{4 w, 1 b} \to \text{box 2} \\ \text{4 w, 6 b} \to \text{box 3} \end{cases}$

X : the chosen ball is black

$P(X) = ?$

A_1 : we choose box 1

A_2 : we choose box 2

A_3 : we choose box 3

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

$$P(X) = P(X|A_1)\cdot \underline{P(A_1)} + P(X|A_2)\cdot \underline{P(A_2)} + P(X|A_3)\cdot \underline{P(A_3)} = \frac{3}{5}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3} + \frac{6}{10}\cdot\frac{1}{3} =$$

$X|A_1$: the black ball comes from box 1

$$= \frac{1}{3}\left(\frac{3}{5} + \frac{1}{5} + \frac{3}{5}\right) = \frac{1}{3}\cdot\frac{7}{5} = \frac{7}{15}$$

Bayes' rule

$A_1, A_2, \dots, A_n$ - partition of Ω

$X \subseteq \Omega$ - event

$$P(A_k/X) = \frac{P(X/A_k)\cdot P(A_k)}{P(X)} \overset{LOTP}{=} \frac{P(X|A_k)\cdot P(A_k)}{P(X|A_1)\cdot P(A_1) + P(X/A_2)\cdot P(A_2) + \dots + P(X/A_n)\cdot P(A_n)}$$

$$P(A_k/X) = \frac{P(A_k \cap X)}{P(X)} = \frac{P(X/A_k)\cdot P(A_k)}{P(X)}$$

$$P(A_k \cap X) = P(X|A_k)\cdot P(A_k)$$

$$\downarrow$$

$$P(X \cap A_k) = \underbrace{P(X|A_k)\cdot P(A_k) = P(A_k|X)\cdot P(X)}\, \Leftrightarrow P(A_k|X) = \frac{P(X/A_k)\cdot P(A_k)}{P(X)}$$

Ex2 :: 2 boxes $\begin{cases} 2w, 3b \longrightarrow \text{box } 1 \\ 7w, 5b \longrightarrow \text{box } 2 \end{cases}$

A_1 : we choose box 1

A_2 : we choose box 2

$P(A_1) = 0.4$, $P(A_2) = 0.6$

X : we choose a black ball

$P(A_2 / X) =$ "probability that the chosen black ball come from the second box"

$$P(A_2 / X) = \frac{P(X|A_2) \cdot P(A_2)}{P(X)} = \frac{P(X|A_2) \cdot P(A_2)}{P(X|A_1) \cdot P(A_1) + P(X|A_2) \cdot P(A_2)} = \frac{\frac{5}{12} \cdot 0.6}{\frac{3}{5} \cdot 0.4 + \frac{5}{12} \cdot 0.6} =$$

$$= \frac{\frac{5}{12} \cdot \frac{6}{10}}{\frac{3}{5} \cdot \frac{4}{10} + \frac{5}{12} \cdot \frac{6}{10}} = \frac{\frac{1}{4}}{\frac{6}{25} + \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{49}{100}} = \frac{1}{4} \cdot \frac{100}{49} = \frac{25}{49} = 0.51$$

Ex.: Compute the PMF and CDF of X, given by:

$$X: \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

PMF: $f: \mathbb{R} \to [0,1]$,

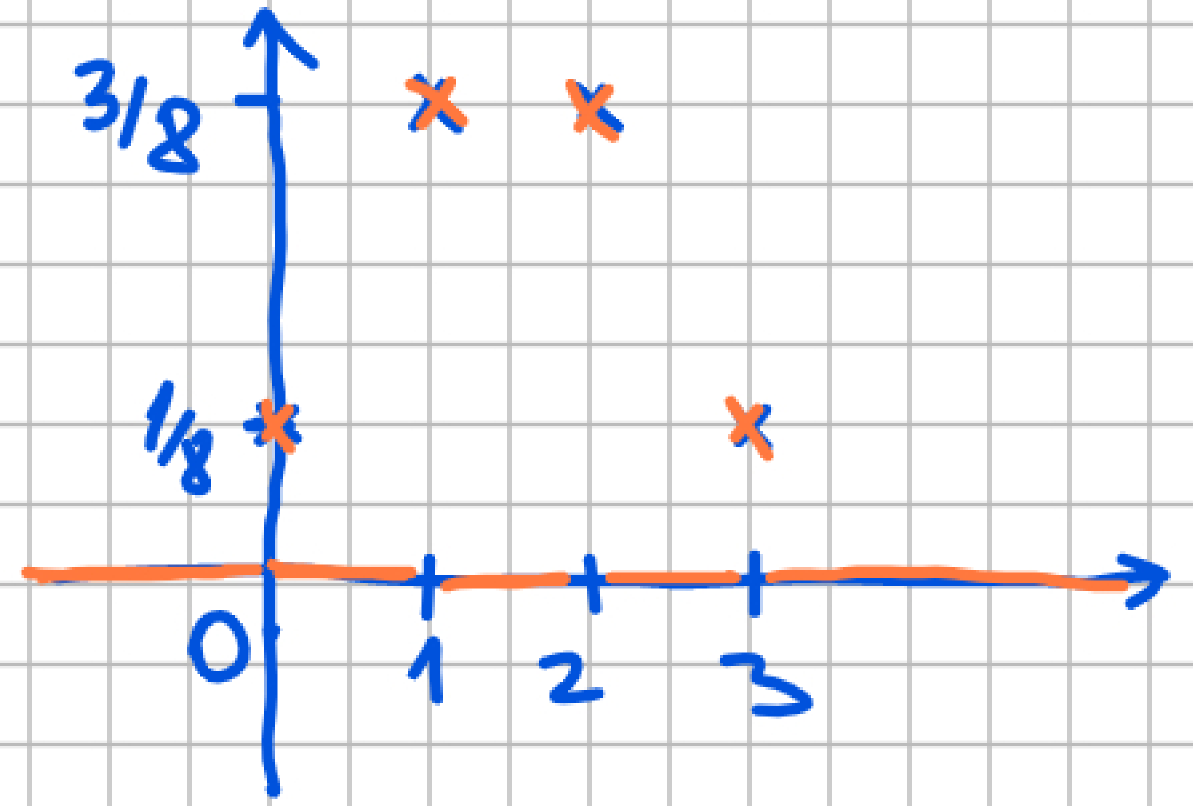
$$f(0) = P(X=0) = \frac{1}{8}$$

$$f(1) = P(X=1) = \frac{3}{8}$$

$$f(2) = P(X=2) = \frac{3}{8}$$

$$f(3) = P(X=3) = \frac{1}{8}$$

$$f(x) = 0, \ (\forall)\, x \notin \{0,1,2,3\}$$



CDF : $F:\mathbb{R}\to[0,1]$,

$$X:\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

$$F(x)=P(X\le x)=0$$

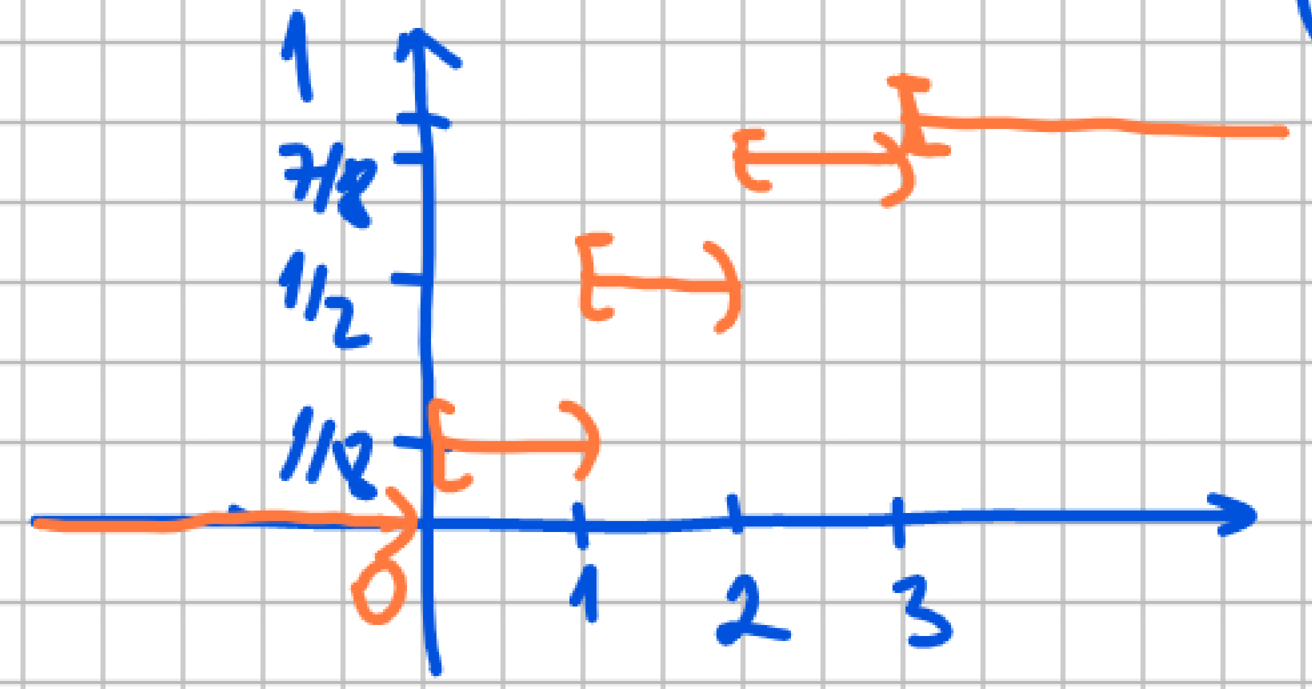
$$F\left(\tfrac{1}{2}\right)=P\left(X\le \tfrac{1}{2}\right)=P(X=0)=\frac{1}{8},$$

$$F\left(\tfrac{3}{2}\right)=P\left(X\le \tfrac{3}{2}\right)=P(X=0)+P(X=1)=$$
$$=\frac{1}{8}+\frac{3}{8}$$

$$F\left(\tfrac{5}{2}\right)=P\left(X\le \tfrac{5}{2}\right)=P(X=0)+P(X=1)+$$
$$+P(X=2)=\frac{1}{8}+2\cdot\frac{3}{8}=\frac{7}{8}$$

$$F(4)=P(X\le 4)$$

$$F(x)=\begin{cases} 0, & x<0 \\ \frac{1}{8}, & x\in[0,1) \\ \frac{1}{8}+\frac{3}{8}, & x\in[1,2) \\ \frac{1}{8}+\frac{3}{8}+\frac{3}{8}, & x\in[2,3) \\ \frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8}, & x\geq 3 \end{cases} = \begin{cases} 0, & x<0 \\ \frac{1}{8}, & x\in[0,1) \\ \frac{1}{2}, & x\in[1,2) \\ \frac{7}{8}, & x\in[2,3) \\ 1, & x\geq 3 \end{cases}$$



Ex: $X: \begin{pmatrix} 5.2 & 5.3 & 5.8 & 6 \\ 0.4 & 0.2 & 0.2 & 0.2 \end{pmatrix}$

$$E[X] = 5.2 \cdot 0.4 + 5.3 \cdot 0.2 + 5.8 \cdot 0.2 + 6 \cdot 0.2 =$$

$$= 2.08 + 1.06 + 1.16 + 1.2 = 3.14 + 2.36 = 5.5$$

$$V[X] = E[X^2] - E^2[X] = 30.362 - 5.5^2 = 0.112 > 0$$

$X^2: \begin{pmatrix} 5.2^2 & 5.3^2 & 5.8^2 & 6^2 \\ 0.4 & 0.2 & 0.2 & 0.2 \end{pmatrix}$, $X^2: \begin{pmatrix} 27.04 & 28.09 & 33.64 & 36 \\ 0.4 & 0.2 & 0.2 & 0.2 \end{pmatrix}$

$$E[X^2] = 27.04 \cdot 0.4 + 28.09 \cdot 0.2 + 33.64 \cdot 0.2 + 36 \cdot 0.2 = 30.362$$

$$Std[X] = \sqrt{V[X]} = \sqrt{0.112} = 0.335$$

$$X^2 : \begin{pmatrix} \overset{=1}{(-1)^2} & 0 & 1^2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad X^2 : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$E[X^2] = \frac{1}{2}$$

$$V[X] = E[X^2] - E^2[X] = \frac{1}{2} - 0 = \frac{1}{2}.$$