Lecture 3. Continuous random variables

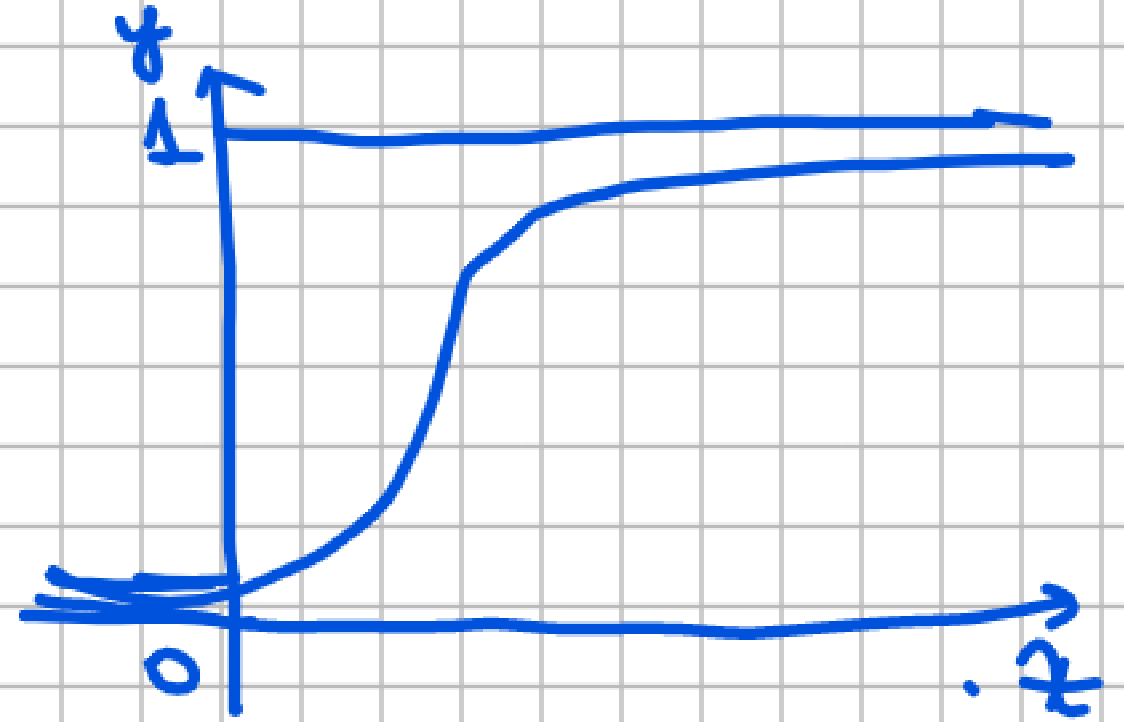
Hypergeometric distribution

30 cards $\to$ 12 red, 18 black

5 cards are drawn without replacement

X - nr. of cards drawn that are red, $X \sim HGeom(12, 18, 5)$

$$P(X=3) = \frac{\binom{12}{3} \cdot \binom{18}{2}}{\binom{30}{5}} = 0.236$$



CDF

$$\int \frac{1}{x^3}\,dx = \frac{x^{-2}}{-2} = -\frac{1}{2}\cdot\frac{1}{x^2}$$

Ex. 1 : X - continuous r.v. with PDF $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases} \frac{k}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases} = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1, \end{cases}$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx$$

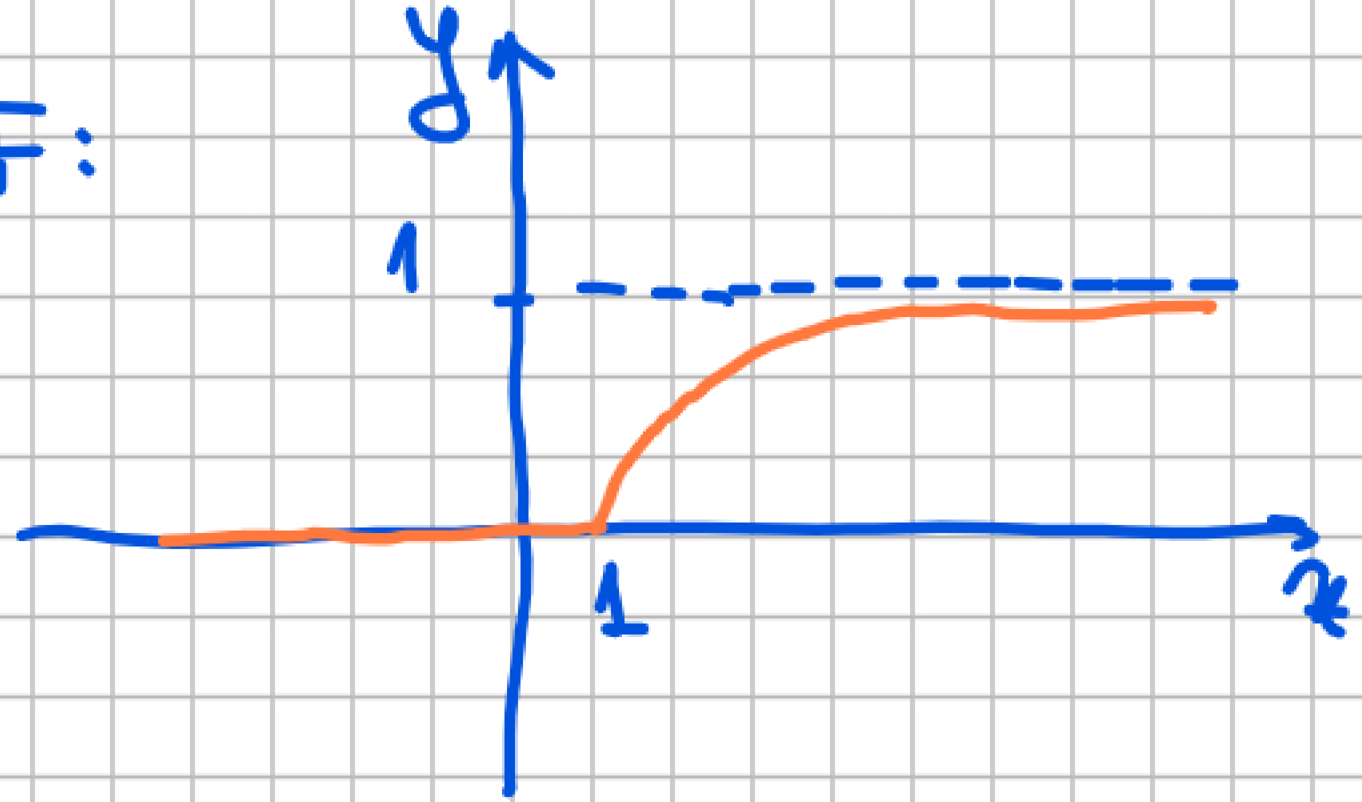
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \underbrace{\int_{-\infty}^{1} f(x)\,dx}_{=0} + \int_{1}^{\infty} f(x)\,dx = 0 + \int_{1}^{\infty} \frac{k}{x^3}\,dx = k\cdot\int_{1}^{\infty}\frac{1}{x^3}\,dx =$$

$$= k\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{x^2}\Big|_1^{\infty} = -\frac{k}{2}\left[0-1\right] = \frac{k}{2}$$

$$\Rightarrow \frac{k}{2} = 1 \iff \boxed{k=2}$$

PDF :

CDF :



CDF : $F(x) = \int_{-\infty}^{x} f(t)\,dt$

- $x \leq 1 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$
- $x > 1 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{1} 0\,dt + \int_{1}^{x} \frac{2}{t^3}\,dt =$

$$= \frac{-1}{t^2}\Big|_1^x = -\left[\frac{1}{x^2} - 1\right] = 1 - \frac{1}{x^2}$$

$$F(x) = \begin{cases} 0, & x \leq 1 \\ 1 - \frac{1}{x^2}, & x > 1 \end{cases}$$

X - lifetime of the electronic component

$$P(X>5) = P(X \in (5,\infty)) = \int_5^\infty f(t)\,dt = \int_5^\infty \frac{2}{x^3}\,dx = \frac{-1}{x^2}\Big|_5^\infty = -\left[0 - \frac{1}{25}\right] = \frac{1}{25} = 0.4$$

Ex. 2 $\quad f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{1} x \cdot 0\,dx + \int_{1}^{\infty} x \cdot \frac{2}{x^3}\,dx = \int_{1}^{\infty} \frac{2}{x^2}\,dx = -\frac{2}{x}\Big|_1^{\infty} = -2(0-1) = 2$$

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E^2[X] =$$

$$= \int_{-\infty}^{1} x^2 \cdot 0\,dx + \int_{1}^{\infty} x^2 \cdot \frac{2}{x^3}\,dx - 4 = \int_{1}^{\infty} \frac{2}{x}\,dx - 4 = 2\ln(x)\Big|_1^{\infty} - 4 = 2(\infty - 0) - 4 = \infty$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{1}^{\infty} x^2 \cdot \frac{2}{x^3}\,dx = \int_{1}^{\infty} \frac{2}{x}\,dx = 2\ln(x)\Big|_1^{\infty} = \infty$$

$$E[\sqrt{X}] = \int_{1}^{\infty} \sqrt{x}\, f(x)\,dx = \int_{1}^{\infty} \sqrt{x} \cdot \frac{2}{x^3}\,dx = \int_{1}^{\infty} 2 \cdot x^{\frac{1}{2}-3}\,dx = 2\int_{1}^{\infty} x^{-5/2}\,dx = 2 \cdot \frac{x^{-3/2}}{-3/2}\Big|_1^{\infty} = -\frac{4}{3}(0-1) = \frac{4}{3}$$

In general: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$; $V[g(X)] = \int_{-\infty}^{\infty} g^2(x) f(x)\,dx - E^2[g(X)]$

Ex. 3: $X \sim U(a,b)$, $f(x) = \frac{1}{b-a}$, $(\forall)\, x \in (a,b)$

$$E[X] = \int_a^b x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{1}{2(b-a)}(b^2-a^2) = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

$$V[X] = \int_a^b x^2 f(x)\,dx - E^2[X] = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx - \frac{(a+b)^2}{4} = \frac{1}{b-a} \cdot \frac{x^3}{3}\Big|_a^b - \frac{(a+b)^2}{4} =$$

$$= \frac{1}{3(b-a)}(b^3-a^3) - \frac{(a+b)^2}{4} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} - \frac{(a+b)^2}{4} = {}^{4)}\frac{a^2+ab+b^2}{3} - {}^{3)}\frac{a^2+2ab+b^2}{4} =$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

Ex. 4: X - arrival time of the flight

$$X \sim U(4{:}50,\, 5{:}10) = U(290, 310) \rightarrow \text{minutes}$$

$$P(X \geq 5{:}05) = P(X \geq 305) = \int_{305}^{310} f(x)\,dx = \int_{305}^{310} \frac{1}{310-290}\,dx = \frac{1}{20}\int_{305}^{310} 1\,dx = \frac{1}{20} \cdot x\Big|_{305}^{310} = \frac{5}{20} = \frac{1}{4} = 0.25$$

Ex.7 : X – time between jobs sent to a printer

$X \sim Exp(\lambda)$, $f(x) = \lambda e^{-\lambda x}$, $x > 0$

a) $\lambda = 3$ (jobs/h)

b) $P(X \leq 5 \text{ min}) = P(X \leq \frac{5}{60} h) = P(X \leq \frac{1}{12}) = \int_0^{\frac{1}{12}} 3e^{-3x}\,dx = -e^{-3x}\Big|_0^{1/12} =$

$= 1 - e^{-1/12} = 0.08$