

Lecture 2. Families of discrete distributions.

Binomial distribution

- n Bernoulli trials
- probability of a "success" is p

X - represents number of "successes" in n trials
 (nr. of "heads" in n tosses of a coin)

$$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ \dots & \dots & \dots & \dots & \binom{n}{k} p^k (1-p)^{n-k} & \dots & \dots \end{pmatrix}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{0}{1} \frac{1}{2} \frac{1}{3} \dots \frac{1}{n-1} \frac{0}{n} \rightarrow p^k (1-p)^{n-k}$$

$$X = X_1 + X_2 + \dots + X_m$$

X_i - Bernoulli random variable with parameter p

$$P(X_i = 1) = p, \quad X_i: \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}, \quad i = \overline{1, n}$$

X_i - independent r.v.s.

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_m] = m \cdot p$$

$$E[X_i] = p$$

$$V[X] = V[X_1] + V[X_2] + \dots + V[X_m] = mp(1-p) = mpq$$

$$V[X_i] = E[X_i^2] - E^2[X_i] = p - p^2 = p(1-p)$$

$$1-p \stackrel{m}{=} q$$

$$\text{Ex. 4: } X_i: \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \quad i=1, \dots, 10$$

$$p = 20\%, \quad p = 0.2$$

X - nr. of new subscribers that get the special promotion (out of the 10)

"success" - getting the promotion

$$X = X_1 + X_2 + \dots + X_{10}$$

$$P(X \geq 4) = P(X=4) + P(X=5) + \dots + P(X=10)$$

$$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & 10 \\ & & & & \binom{10}{k} p^k (1-p)^{10-k} & & \dots \end{pmatrix}$$

$$X \sim \text{Bin}(n, p)$$

$$n=10, \quad p=0.2$$

$$P(X=k) = \binom{10}{k} p^k (1-p)^{10-k}$$

$$\frac{\binom{10}{3} \cdot 0.2^3 \cdot 0.8^7}{1} = 120$$

$$P(X \geq 4) = 1 - P(X < 4) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)] =$$

$$= 1 - \left[\binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p^1 (1-p)^9 + \binom{10}{2} p^2 (1-p)^8 + \binom{10}{3} p^3 (1-p)^7 \right] =$$

$$= 1 - [0.8^{10} + 10 \cdot 0.2 \cdot 0.8^9 + 45 \cdot 0.2^2 \cdot 0.8^8 + 120 \cdot 0.2^3 \cdot 0.8^7] = 0.121$$

Ex. 5: X - nr. of users (among the 15) that buy the advanced version

$$X \sim \text{Bin}(n, p)$$

$$n = 15$$

$$p = 0.6 \cdot 0.3 = 0.18$$

p - probab. that an user buys the advanced version

$$E[X] = ? , E[X] = n \cdot p(1-p) = 15 \cdot 0.18 \cdot 0.82 = 2.214$$

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = 1 - \left[\binom{15}{0} p^0 (1-p)^{15} + \binom{15}{1} p^1 (1-p)^{14} \right] = \\ &= 1 - [0.82^{15} + 15 \cdot 0.18 \cdot 0.82^{14}] = 0.781 \end{aligned}$$

$$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & 15 \\ \dots & \dots & \binom{15}{k} p^k (1-p)^{15-k} & \dots & \dots & \dots & \dots \end{pmatrix}$$

Geometric distribution

X : nr. of Bernoulli trials needed to get the first success

$$X: \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ \dots & \dots & \dots & \dots & (1-p)^{k-1} \cdot p & \dots \end{pmatrix} \quad \begin{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 3 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ k \end{bmatrix} & \begin{bmatrix} 0 \\ \infty \end{bmatrix} \end{matrix}$$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$E[X] = 1 \cdot (1-p)^0 \cdot p + 2 \cdot (1-p)^1 \cdot p + \dots + k \cdot (1-p)^{k-1} \cdot p + \dots = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \frac{1}{p}$$

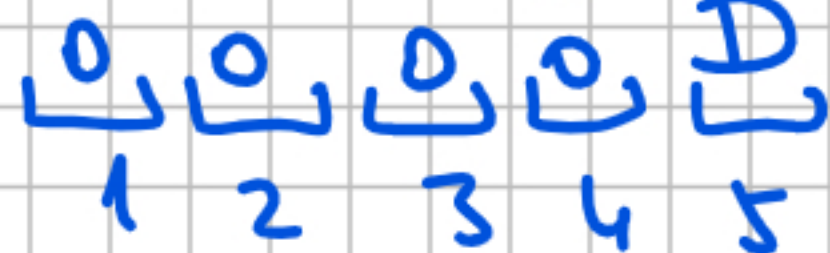
$$V[X] = \frac{1-p}{p^2}$$

X' : nr. of failures before the first success

$$X': \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ \dots & \dots & \dots & \dots & (1-p)^k \cdot p & \dots \end{pmatrix}$$

Ex 6 a) X - nr. of items inspected until we find a defective one

$$P(X=5) = ?$$



$$P(X=5) = (1-p)^4 \cdot p = 0.97^4 \cdot 0.03 = 0.027,$$

$$X \sim \text{Geom}(p)$$

p = probab. that an item is defective

$$p = 0.03$$

b) X - nr. of terminals polled until we find the first one that is ready to transmit

$$X \sim \text{Geom}(p)$$

$$p = 0.95$$

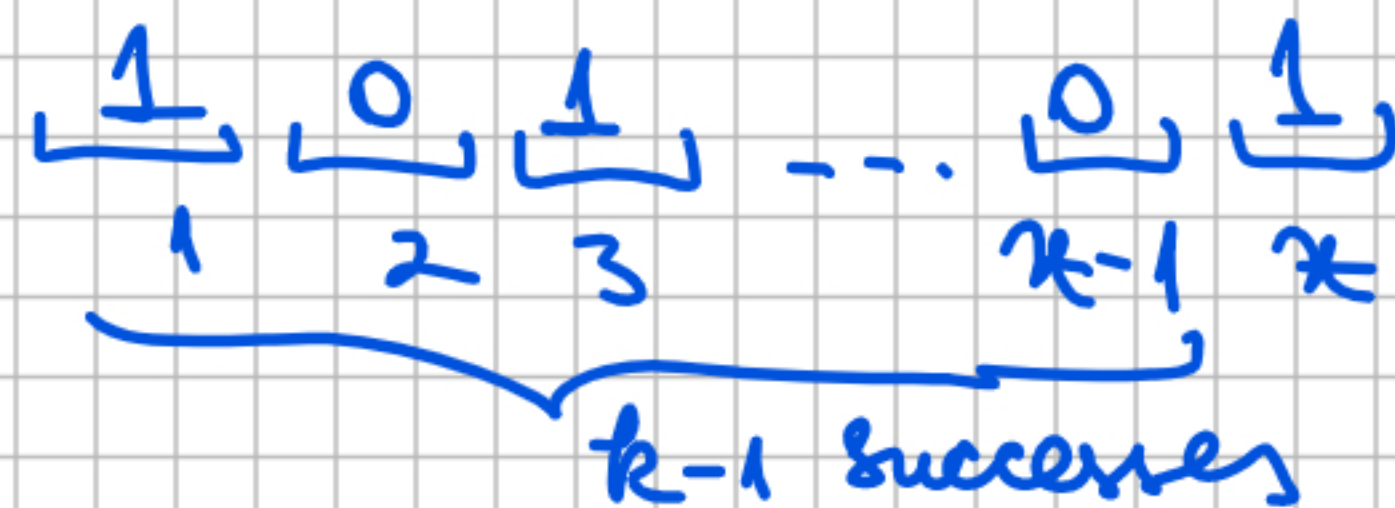
$$P(X=3) = ?, P(X \geq 3) = 1 - P(X < 3), E[X] = ?$$

Negative Binomial distribution

X - nr. of trials needed to get k successes

$$X \sim NB(k, p)$$

$$X: (k \quad k+1 \quad k+2 \quad \dots \quad x \quad \dots)$$



$$P(X=x) = \binom{x-1}{k-1} \cdot p^k (1-p)^{x-k}$$

Ex. 8: X - nr. of new accounts in a day

$$X \sim P_0(\lambda)$$

$$\lambda = 10$$

$$a) P(X > 8) = 1 - P(X \leq 8) = 1 - [P(X=0) + P(X=1) + \dots + P(X=8)] =$$

$$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= 1 - \left[e^{-\lambda} \cdot 1 + e^{-\lambda} \cdot \frac{\lambda^1}{1!} + e^{-\lambda} \cdot \frac{\lambda^2}{2!} + \dots + e^{-\lambda} \cdot \frac{\lambda^8}{8!} \right] =$$

$$= 1 - e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \dots + \frac{\lambda^8}{8!} \right] = 0.667,$$

b) Y - nr. of new accounts in a period of 2 days

$$Y \sim P_0(\lambda'), \quad \lambda' = 20$$

$$P(Y > 16) = 1 - P(Y \leq 16) = \dots$$