Lecture 2. Families of discrete distributions.

Binomial distribution

- n Bernoulli trials
- probability of a "success" is p

X - represents number of "successes" in n trials
(nr. of "Heads" in n tosses of a coin)

$$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & \dots & & & \binom{n}{k} p^k (1-p)^{n-k} & \dots & \end{pmatrix}$$

$$\underset{1}{\underline{0}} \; \underset{2}{\underline{1}} \; \underset{3}{\underline{1}} \; \dots \; \underset{n-1}{\underline{1}} \; \underset{n}{\underline{0}} \longrightarrow p^k (1-p)^{n-k}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$X = X_1 + X_2 + \dots + X_m$$

X_i – Bernoulli random variable with parameter p

$P(X_i = 1) = p$, $X_i : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$, $i = \overline{1,n}$

X_i – independent r.v.s.

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_m] = m \cdot p$$

$$E[X_i] = p$$

$$V[X] = V[X_1] + V[X_2] + \dots + V[X_m] = mp(1-p) = mpq$$

$$V[X_i] = E[X_i^2] - E^2[X_i] = p - p^2 = p(1-p)$$

$$1 - p \overset{M}{=} q$$

Ex. 4 : $X_i : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$ $i=\overline{1,10}$

"success" - getting the promotion

$p = 20\%$, $p=0.2$

$X = X_1 + X_2 + \dots + X_{10}$

X - nr. of new subscribers that get the special promotion (out of the 10)

$P(X \geq 4) = P(X=4) + P(X=5) + \dots + P(X=10)$

$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & 10 \\ & \dots & & \binom{10}{k} p^k (1-p)^{10-k} & \dots & & \end{pmatrix}$

$X \sim Bin(n,p)$

$n=10$, $p=0.2$

$P(X=k) = \binom{10}{k} p^k (1-p)^{10-k}$

$\frac{10 \cdot 9 \cdot 8}{6} = 120$

$P(X \geq 4) = 1 - P(X<4) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)] =$

$= 1 - \left[\underbrace{\binom{10}{0}}_{=1} p^0 (1-p)^{10} + \underbrace{\binom{10}{1}}_{=10} p^1 (1-p)^9 + \underbrace{\binom{10}{2}}_{=45} p^2 (1-p)^8 + \underbrace{\binom{10}{3}}_{=120} p^3 (1-p)^7 \right] =$

$= 1 - [0.8^{10} + 10 \cdot 0.2 \cdot 0.8^9 + 45 \cdot 0.2^2 \cdot 0.8^8 + 120 \cdot 0.2^3 \cdot 0.8^7] = 0.121$

Ex. 5: X – nr. of users (among the 15) that buy the advanced version

$X \sim Bin(n, p)$

$n = 15$

$p = 0.6 \cdot 0.3 = 0.18$ //

$X : \begin{pmatrix} 0 & 1 & 2 & \dots k \dots & 15 \\ & \dots & \binom{15}{k} p^k (1-p)^{15-k} & & \dots \end{pmatrix}$

p – probab. that an user buys the advanced version

$E[X] = ?$, $E[X] = n \cdot p(1-p) = 15 \cdot 0.18 \cdot 0.82 = 2.214$

$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = 1 - [\underbrace{\binom{15}{0}}_{1} p^0 (1-p)^{15} + \underbrace{\binom{15}{1}}_{15} p^1 (1-p)^{14}] =$

$= 1 - [0.82^{15} + 15 \cdot 0.18 \cdot 0.82^{14}] = 0.781$

Geometric distribution

X : nr. of Bernoulli trials needed to get the first success

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$$

$$\underbrace{0}_{1} \quad \underbrace{0}_{2} \quad \underbrace{0}_{3} \quad \dots \underbrace{0}_{k-1} \; \underbrace{\overset{S}{1}}_{k}$$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$E[X] = 1 \cdot (1-p)^{0} \cdot p + 2 \cdot (1-p)^{1} \cdot p + \dots + k(1-p)^{k-1} \cdot p + \dots = \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p = \frac{1}{p}$$

$$V[X] = \frac{1-p}{p^2}$$

X' : nr. of failures before the first success

$$X' : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & \dots & & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$$

Ex6 a) X - nr. of items inspected until we find a defective one

$P(X=5)=?$

$$\underbrace{O}_{1}\ \underbrace{O}_{2}\ \underbrace{O}_{3}\ \underbrace{O}_{4}\ \underbrace{\overset{1}{D}}_{5}$$

$$P(X=5)=(1-p)^4\cdot p=0.97^4\cdot 0.03=0.027$$

$X\sim Geom(p)$

p = probab. that an item is defective

$p=0.03$

b) X - nr. of terminals polled until we find the first one that is ready to transmit

$X\sim Geom(p)$

$p=0.95$

$P(X=3)=?$, $P(X\geq 3)=1-P(X<3)$, $E[X]=?$

Negative Binomial distribution

X - nr. of trials needed to get k successes

$X \sim NB(k, p)$

$X : \begin{pmatrix} k & k+1 & k+2 & \dots & x & \dots \\ & & & & & \end{pmatrix}$

$$\underbrace{\underset{1}{\underline{1}} \ \underset{2}{\underline{0}} \ \underset{3}{\underline{1}} \ \dots \ \underset{x-1}{\underline{0}}}_{k-1 \text{ successes}} \ \underset{x}{\underline{1}}$$

$$P(X = x) = \binom{x-1}{k-1} \cdot p^k (1-p)^{x-k}$$

Ex. 8: X - nr. of new accounts in a day

$X \sim Po(\lambda)$

$\lambda = 10$

a) $P(X>8) = 1 - P(X \le 8) = 1 - [P(X=0) + P(X=1) + \dots + P(X=8)] =$

$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$= 1 - \left[e^{-\lambda} \cdot 1 + e^{-\lambda} \cdot \frac{\lambda^1}{1!} + e^{-\lambda} \cdot \frac{\lambda^2}{2!} + \dots + e^{-\lambda} \cdot \frac{\lambda^8}{8!}\right] =$

$= 1 - e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \dots + \frac{\lambda^8}{8!}\right] = 0.667$

b) Y - nr. of new accounts in a period of 2 days

$Y \sim Po(\lambda')$, $\lambda' = 20$

$P(Y>16) = 1 - P(Y \le 16) = \dots$