Lab 6. Continuous random variables. Simulations in R.

sgr. 3

7. Let

$$F(x) = \frac{2}{\pi}\sin^{-1}\left(\sqrt{x}\right), \text{ for } 0 < x < 1,$$

$F(x) = 0$ for $x \leq 0$, and $F(x) = 1$ for $x \geq 1$.

(a) Check that F is a valid CDF, and find the corresponding PDF f. This distribution is called the *Arcsine distribution*, though it also goes by the name Beta(1/2, 1/2) (we will explore the Beta in depth in Chapter 8).

(b) Explain how it is possible for f to be a valid PDF even though $f(x)$ goes to ∞ as x approaches 0 from the right and as x approaches 1 from the left.

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{2}{\pi}\arcsin(\sqrt{x}), & x \in (0,1) \\ 1, & x \geq 1 \end{cases}$$

a) F - nondecreasing ✓

F - continuous (differentiable) ✓

$\lim\limits_{x\to-\infty} F(x) = 0$, $\lim\limits_{x\to\infty} F(x) = 1$ ✓

$\lim\limits_{\substack{x\to 0\\ x<0}} F(x) = 0$, $\lim\limits_{\substack{x\to 0\\ x>0}} F(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{2}{\pi}\arcsin(\sqrt{x}) = 0 \Rightarrow$ F-continuous at $x=0$

$\lim\limits_{\substack{x\to 1\\ x<1}} F(x) = \lim\limits_{\substack{x\to 1\\ x<1}} \frac{2}{\pi}\arcsin(\sqrt{x}) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$, $\lim\limits_{\substack{x\to 1\\ x>1}} F(x) = 1 \Rightarrow$ F-continuous at $x=1$

$\Rightarrow F$ is a valid CDF

PDF : $f(x) = F'(x)$

$$f(x) = \begin{cases} 0 & , x \le 0 \\ \dfrac{1}{\pi\sqrt{x(1-x)}} & , x \in (0,1) \\ 0 & , x \ge 1 \end{cases}$$

$$\left(\frac{2}{\pi}\arcsin(\sqrt{x})\right)' = \frac{\cancel{2}}{\pi}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{\cancel{2}\sqrt{x}} = \frac{1}{\pi}\cdot\frac{1}{\sqrt{x(1-x)}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{cases} f(x) \ge 0, \ (\forall)\, x \in \mathbb{R} \\ f\text{-integrable} \\ \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \lim_{a\to\infty}\int_{-a}^{a} f(x)\,dx = \lim_{a\to\infty}\left(F(a) - F(-a)\right) = 1 \end{cases}$$

$$\lim_{\substack{x\to 0\\ x<0}} \frac{F(x)-F(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{0}{x} = 0$$

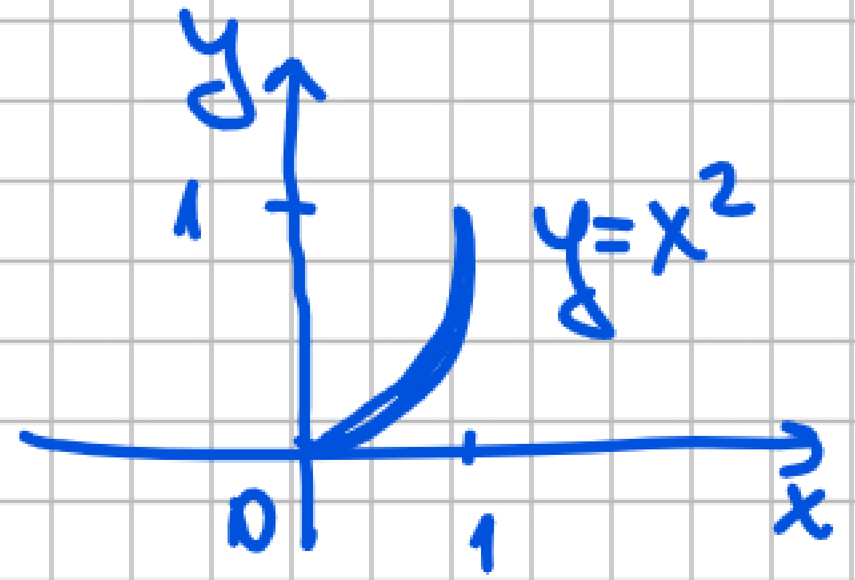
$$\lim_{\substack{x\to 0\\ x>0}} \frac{F(x)-F(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{2}{\pi}\cdot\frac{\arcsin(\sqrt{x})}{x} \overset{l'H}{=}$$

$$= \frac{2}{\pi}\lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\sqrt{x(1-x)}}}{1} = \frac{2}{\pi}\lim_{\substack{x\to 0\\ x>0}} \frac{1}{\sqrt{x(1-x)}} = \infty$$

$$\lim_{\substack{x\to 1\\ x<1}} \frac{F(x)-F(1)}{x-1} = \lim_{\substack{x\to 1\\ x<1}} \frac{\frac{2}{\pi}\arcsin(\sqrt{x})-1}{x-1} \overset{l'H}{=}$$

$$= \lim_{\substack{x\to 1\\ x<1}} \frac{1}{\pi\sqrt{x(1-x)}} = \infty$$

2. Let $F:\mathbb{R}\to\mathbb{R}$, $F(x)=\begin{cases} 0, & x\le 0 \\ ax^2, & x\in(0,1] \\ 1, & x>1 \end{cases}$



a) Find $a\in\mathbb{R}$ such that F is a valid CDF.

b) Compute $f(x)$ (PDF).

c) Compute $E[X]$ and $V[X]$.

d) Find $P\left(\frac{1}{4}<X<\frac{3}{4}\right)$.

a) F - nondecreasing $\iff a>0$

F - continuous $\iff \lim\limits_{\substack{x\to 1\\ x\le 1}} F(x)=a$, $\lim\limits_{\substack{x\to 1\\ x>1}} F(x)=1 \implies \boxed{a=1}$

$\lim\limits_{x\to-\infty} F(x)=0$, $\lim\limits_{x\to\infty} F(x)=1$ ✓

b) $f(x)=2x,\ x\in(0,1)$, $f(x)=0$ on $\mathbb{R}\setminus(0,1)$

c) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \int_0^1 2x^2\,dx = 2\cdot \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$

$V[X] = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E^2[X] = \int_0^1 x^2 \cdot 2x\,dx - \frac{4}{9} = \int_0^1 2x^3\,dx - \frac{4}{9} = \cancel{2}\cdot \frac{x^4}{\cancel{4}_{2}}\Big|_0^1 - \frac{4}{9} =$

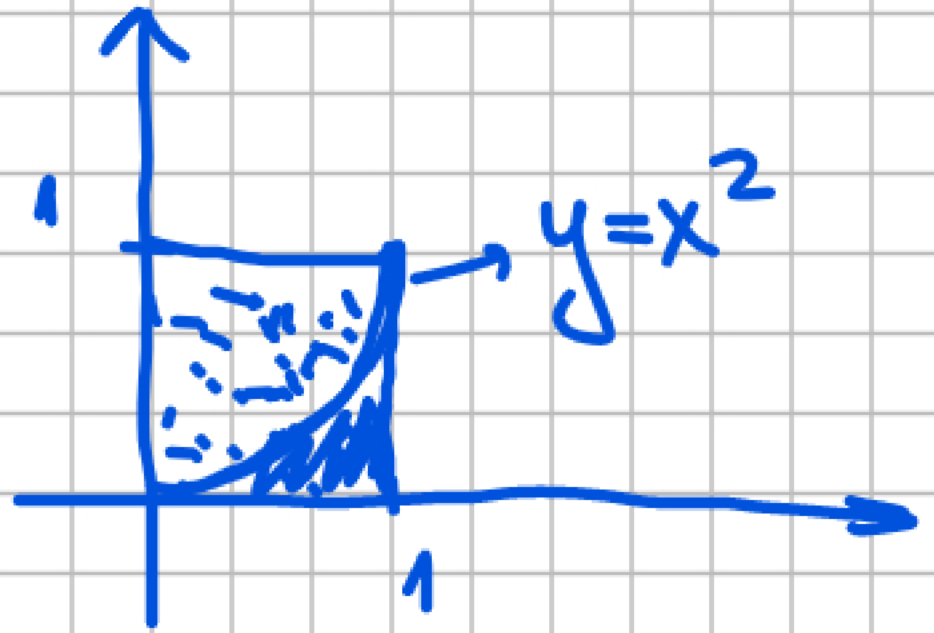
$= \frac{1}{2}^{9} - \frac{4}{9}^{2} = \frac{1}{18}$ //

d) $P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} f(x)\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} 2x\,dx = x^2 \Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{9}{16} - \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$ //

lab4.pdf

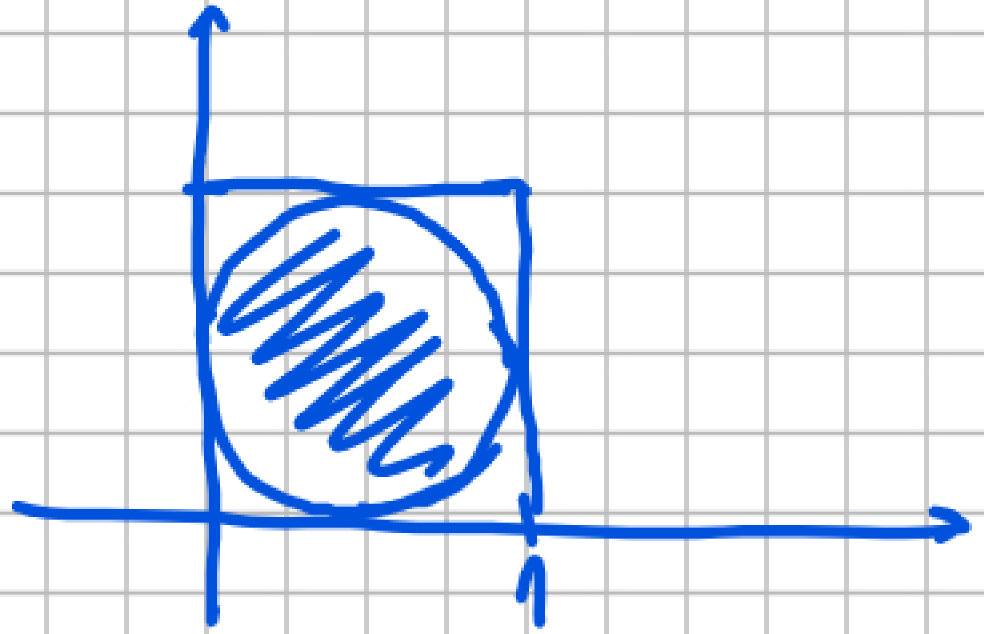
A1. Monte Carlo method

a)



$y < x^2$

b)



$$E[X] = \int_0^\infty \lambda \cdot x e^{-\lambda x} dx = \int_0^\infty 7 x e^{-7x} dx$$

Homework: assignment 4 from lab4.pdf

sgr. 2

8. The *Beta distribution* with parameters $a = 3, b = 2$ has PDF

$$f(x) = 12x^2(1-x), \text{ for } 0 < x < 1.$$

(We will discuss the Beta in detail in Chapter 8.) Let X have this distribution.

(a) Find the CDF of X.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

$$f(x) = \begin{cases} 0, & x \le 0 \\ 12x^2(1-x), & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

$\Rightarrow f \in \mathcal{C}_{\mathbb{R}}$

$f(x) \geq 0, (\forall) x \in \mathbb{R}$

f - is a valid PDF: $f(x) \geq 0$ ✓

$f(x) = F'(x)$ $\qquad \int_{-\infty}^{\infty} f(x)dx = 1$ ✓

$\lim_{\substack{x \to 0 \\ x < 0}} f(x) = 0$, $\lim_{\substack{x \to 0 \\ x > 0}} f(x) = \lim_{\substack{x \to 0 \\ x > 0}} 12x^2(1-x) = 0$, $f(0) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0 = f(0) \Rightarrow f \in \mathcal{C}_{\{0\}}$

$\lim_{\substack{x \to 1 \\ x < 1}} f(x) = \lim_{\substack{x \to 1 \\ x < 1}} 12x^2(1-x) = 0$, $\lim_{\substack{x \to 1 \\ x > 1}} f(x) = 0$, $f(1) = 0 \Rightarrow f \in \mathcal{C}_{\{1\}}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_0^1 f(x)\,dx + \underbrace{\int_1^{\infty} f(x)\,dx}_{0} = \int_0^1 12x^2(1-x)\,dx =$$

$$= \int_0^1 12x^2\,dx - \int_0^1 12x^3\,dx = \cancel{12}^{4}\cdot\frac{x^3}{\cancel{3}}\Big|_0^1 - \cancel{12}^{3}\cdot\frac{x^4}{\cancel{4}}\Big|_0^1 = 4-3=1$$

$\Rightarrow f$ is a valid PDF

a) CDF: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x \le 0 \\ \int_{-\infty}^{x} 12t^2(1-t)\,dt, & x\in(0,1) \\ 1, & x \ge 1 \end{cases} = \begin{cases} 0, & x\le 0 \\ x^3(4-3x), & x\in(0,1) \\ 1, & x\ge 1 \end{cases}$

$$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{1} f(t)\,dt + \int_1^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x 0\,dt = \int_0^1 f(t)\,dt = 1$$

$$\int_0^x 12t^2(1-t)\,dt = 12\int_0^x t^2\,dt - 12\int_0^x t^3\,dt = \cancel{12}^{4}\cdot\frac{t^3}{\cancel{3}}\Big|_0^x - \cancel{12}^{3}\cdot\frac{t^4}{\cancel{4}}\Big|_0^x = 4x^3 - 3x^4$$

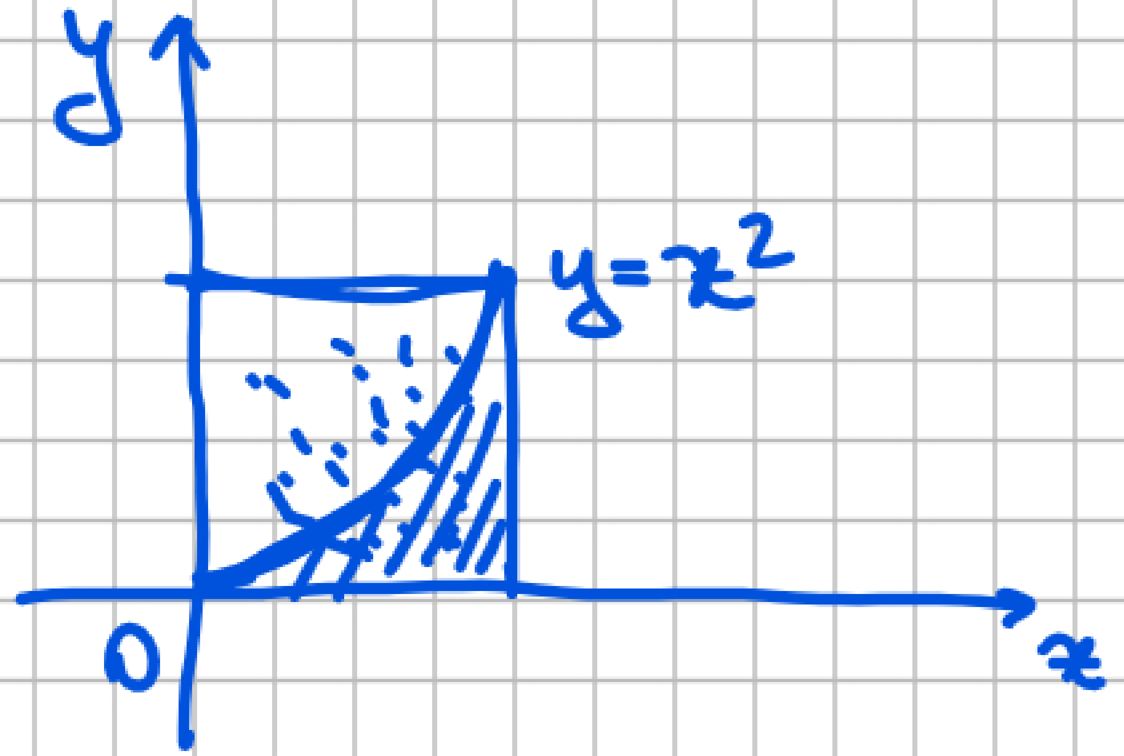
b) $P(0<X<\frac{1}{2}) = \int_0^{1/2} f(x)\,dx = F(\frac{1}{2}) - F(0) = F(\frac{1}{2}) = \frac{1}{2^3}(4-\frac{3}{2}) = \frac{1}{8}\cdot\frac{5}{2} = \frac{5}{16} = 0.313$

c) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\cdot 12x^2(1-x)\,dx = \int_0^1 (12x^3-12x^4)\,dx = 12\cdot\frac{x^4}{4}\Big|_0^1 - 12\cdot\frac{x^5}{5}\Big|_0^1 = \frac{\overset{3}{\cancel{12}}}{\cancel{4}} - \frac{12}{5} = \frac{3}{5} = 0.6$

$V[X] = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E^2[X] = \int_0^1 x^2\cdot 12x^2(1-x)\,dx - 0.6^2 = \int_0^1 (12x^4-12x^5)\,dx - 0.6^2 =$

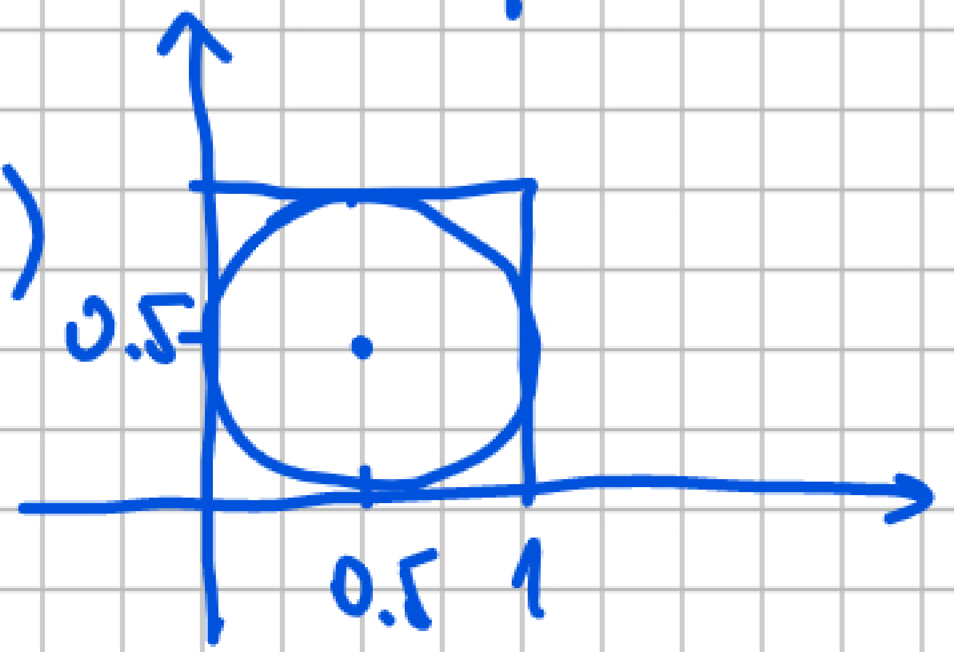
$= 12\cdot\frac{x^5}{5}\Big|_0^1 - \overset{2}{\cancel{12}}\cdot\frac{x^6}{\cancel{6}}\Big|_0^1 - 0.36 = \frac{12}{5} - 2 - 0.36 = 0.4 - 0.6^2 = 0.04$

A1. a)



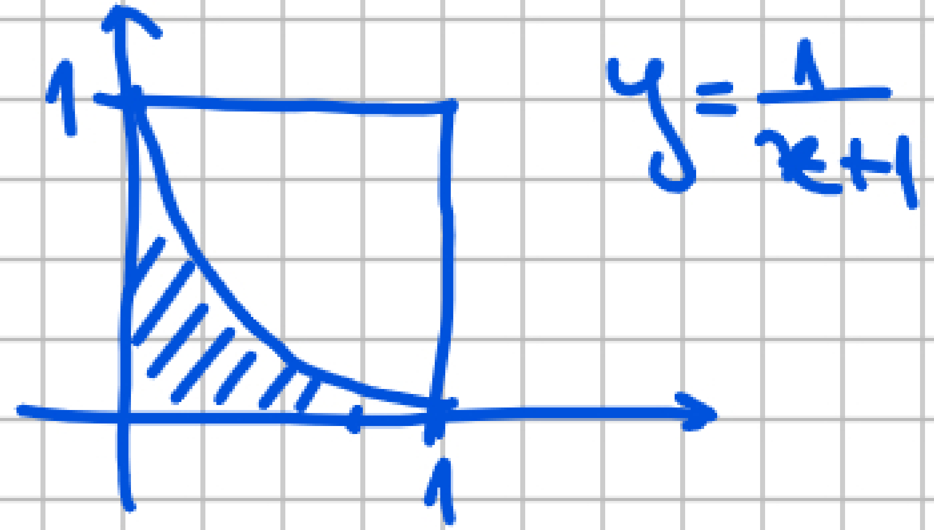
$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} = 0.33$$

b)



$$\text{Area} = \pi \cdot r^2 \Rightarrow \pi \cong \text{area}/0.5^2$$

c)



$$\int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln 2$$

13. Write a function in R with n and λ:

- generates n values in $(0,1)$ → runif → u
- returns $-\frac{\ln(1-u)}{\lambda}$

Homework: assignments 4 and 5 from lab4.pdf.

sgr. 6

9. The *Cauchy distribution* has PDF

$$f(x) = \frac{1}{\pi(1+x^2)},$$

for all real x. (We will introduce the Cauchy from another point of view in Chapter 7.) Find the CDF of a random variable with the Cauchy PDF.

1) Check that f is a valid PDF.

$$\begin{cases} f(x) \geq 0,\ (\forall)\ x \in \mathbb{R} \ \checkmark \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \ \checkmark \\ f\text{-integrable} \ \checkmark \end{cases}$$

$$f(x) = \frac{1}{\pi(1+x^2)} \geq 0,\ (\forall)\ x \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\cdot \operatorname{arctg}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{\pi}\cdot\pi = 1$$

$$X : \begin{pmatrix} x_1 & x_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$$

$$\sum p_i = 1$$

$$\left(\ln(u(x))\right)' = \frac{u'(x)}{u(x)}$$

$$\lim_{x \to \infty} \operatorname{arctg}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \operatorname{arctg}(x) = -\frac{\pi}{2}$$

f is continuous on $\mathbb{R}$ $\Rightarrow$ f is integrable

CDF : $F:\mathbb{R}\to\mathbb{R}$, $\underline{F(x)} = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+t^2}\,dt =$

$= \frac{1}{\pi}\,\mathrm{arctg}(t)\Big|_{-\infty}^{x} = \frac{1}{\pi}\left(\mathrm{arctg}(x)+\frac{\pi}{2}\right) = \frac{1}{\pi}\,\mathrm{arctg}(x)+\frac{1}{2}$

$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x\cdot\frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx = \frac{1}{2\pi}\ln(1+x^2)\Big|_{-\infty}^{\infty}$

$\left(\ln(1+x^2)\right)' = \frac{1}{1+x^2}\cdot 2x$

$\int_{-a}^{a} x f(x)\,dx = \frac{1}{2\pi}\ln(1+x^2)\Big|_{-a}^{a} = \frac{1}{2\pi}\left(\ln(1+a^2)-\ln(1+a^2)\right) = 0$

$\Rightarrow \lim_{a\to\infty}\int_{-a}^{a} x f(x)\,dx = 0 \Leftrightarrow \int_{-\infty}^{\infty} x f(x)\,dx = 0 \rightarrow E[X]=0$

$$V[X] = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \underbrace{E^2[X]}_{0} = \int_{-\infty}^{\infty} \frac{x^2}{\pi(1+x^2)}\,dx$$

$$\int_{-a}^{a} \frac{x^2}{\pi(1+x^2)}\,dx \underset{\uparrow}{=} 2\int_0^a \frac{x^2}{\pi(1+x^2)}\,dx = \frac{2}{\pi}\int_0^a \frac{x^2+1-1}{1+x^2}\,dx = \frac{2}{\pi}\int_0^a 1\,dx - \frac{2}{\pi}\int_0^a \frac{1}{1+x^2}\,dx$$

even function $(g(x)=g(-x))$

$$= \frac{2}{\pi}a - \frac{2}{\pi}\operatorname{arctg} a$$

$$\int_{-\infty}^{\infty} \frac{x^2}{\pi(1+x^2)}\,dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{x^2}{\pi(1+x^2)}\,dx = \lim_{a\to\infty}\left(\frac{2}{\pi}a - \frac{2}{\pi}\underbrace{\operatorname{arctg} a}_{\downarrow\,\frac{\pi}{2}}\right) = \frac{2}{\pi}\cdot\infty - 1 = \infty$$

$$\Rightarrow V[X] = \infty$$

$$P(X\in(0,1)) = \int_0^1 f(x)\,dx = F(1) - F(0) = \frac{1}{\pi}\operatorname{arctg}(1) + \cancel{\frac{1}{2}} - \left(\frac{1}{\pi}\operatorname{arctg}(0) + \cancel{\frac{1}{2}}\right) =$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{4} - \frac{1}{\pi}\cdot 0 = \frac{1}{4}$$

② Consider $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} a\cdot x\cdot e^{-3x}, & x\geq 0 \\ 0, & x<0 \end{cases}$ $\quad a\in\mathbb{R}$

Check that f is a valid PDF. Compute the CDF, $E[X]$, $V[X]$ and $P(0<X<2)$.

$f(x)\geq 0,\ (\forall)\, x\in\mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_{0}^{\infty} f(x)\,dx = \int_0^\infty a x e^{-3x}\,dx = \int_0^\infty a x\left(-\frac{1}{3}e^{-3x}\right)'dx =$$

$$= -\frac{a}{3}xe^{-3x}\Big|_0^\infty + \frac{a}{3}\int_0^\infty e^{-3x}\,dx = \frac{a}{3}\left(-\frac{1}{3}e^{-3x}\right)\Big|_0^\infty = -\frac{a}{9}(0-1) = \frac{a}{9}$$

$\lim\limits_{x\to\infty} xe^{-3x} = \lim\limits_{x\to\infty}\frac{x}{e^{3x}} = 0$,

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \frac{a}{9}=1 \iff \underline{a=9}$

$$f(x) = \begin{cases} 9x e^{-3x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x \leq 0 \\ \int_{-\infty}^{x} 9t e^{-3t}\,dt, & x > 0 \end{cases} = \begin{cases} 0, & x \leq 0 \\ -3xe^{-3x} - e^{-3x} + 1, & x > 0 \end{cases}$$

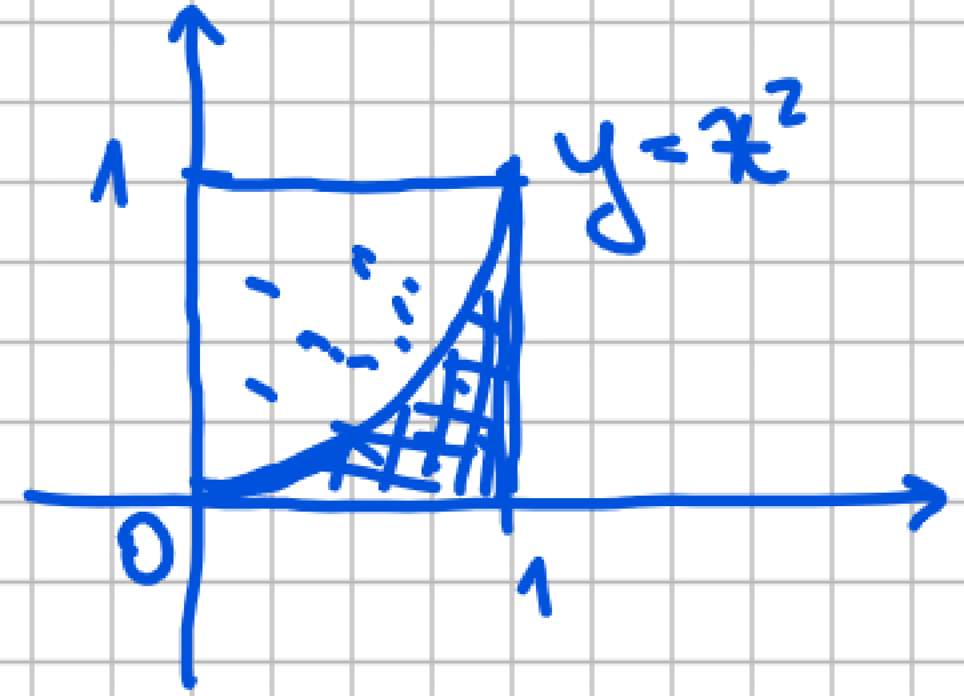
$$\int_{-\infty}^{x} 9t e^{-3t}\,dt = \int_{0}^{x} 9t e^{-3t}\,dt = -3\,t e^{-3t}\Big|_0^x - e^{-3t}\Big|_0^x = -3xe^{-3x} - e^{-3x} + 1$$

$$E[X] = \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} 9x^2 e^{-3x}\,dx = \ldots$$

$$V[X] = \int_0^{\infty} x^2 f(x)\,dx - E^2[X] = \int_0^{\infty} 9x^3 e^{-3x}\,dx - E^2[X] = \ldots$$

$$P(0 < X < 2) = \int_0^2 f(x)\,dx = F(2) - F(0) = F(2) = -6e^{-6} - e^{-6} + 1 = 1 - 7e^{-6} = 0.98$$

A1. Monte Carlo method



$(x,y) \longrightarrow y \leq x^2$

A3. Write a function in R with parameters λ and n:

- generate n random values in $(0,1) \longrightarrow$ runif $\longrightarrow u$
- return $-\log(1-u)/\lambda$

Homework: assignments 2 and 4 from lab4.pdf.

Exercise 79. Bearing Works manufactures bearing shafts whose diameters are normally distributed with parameters $\mu = 1$, $\sigma = .002$. The buyers specifications require these diameters to be $1.000 \pm .003cm$. What fraction of the manufacturers shafts are likely to be rejected? If the manufacturer improves her quality control, she can reduce the value of σ. What value of σ will ensure that no more than 1 percent of her shafts are likely to be rejected?

$1.0 \pm 0.003 \to [0.997, 1.003]$

$\int e^{-x^2} dx$

X - diameter of a bearing shaft

$X \sim N(\mu, \sigma)$ $\quad \mu = 1,\ \sigma = .002$

$$P(X \notin [0.997, 1.003]) = P(X \in (-\infty, 0.997)) + P(X \in (1.003, \infty)) =$$

$$= \int_{-\infty}^{0.997} f(x)\,dx + \int_{1.003}^{\infty} f(x)\,dx = \dots$$

$$= 1 - P(X \in [0.997, 1.003]) = 1 - \int_{0.997}^{1.003} f(x)\,dx = 1 - (F(1.003) - F(0.997))$$

$$= 1 - \text{pnorm}(1.003, 1, 0.002) + \text{pnorm}(0.997, 1, 0.002) = 0.134$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$F(a) = \text{pnorm}(a, \text{mean}=1, \text{sd}=0.002)$

$\Rightarrow$ the fraction of rejected shafts : 13.4%

Probab. of rejected shafts $\le 1\%$

$P(X \notin [0.997, 1.003]) \le 0.01 \iff 1 - \text{pnorm}(1.003, 1, \sigma) + \text{pnorm}(0.997, 1, \sigma) \le 0.01$

$\iff \text{pnorm}(1.003, 1, \sigma) - \text{pnorm}(0.997, 1, \sigma) \ge 0.99 \iff P(X \in [0.997, 1.003]) \ge 0.99$

$$\int_{0.997}^{1.003} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-1)^2}{2\sigma^2}} dx \ge 0.99$$

$$P(0.997 \le X \le 1.003) = P\left(\frac{0.997-1}{\sigma} \le \frac{X-1}{\sigma} \le \frac{1.003-1}{\sigma}\right) = P\left(\frac{-0.003}{\sigma} \le Z \le \frac{0.003}{\sigma}\right) =$$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$$= \phi\left(\frac{0.003}{\sigma}\right) - \phi\left(-\frac{0.003}{\sigma}\right) = 2\phi\left(\frac{0.003}{\sigma}\right) - 1 \ge 0.99 \iff$$

ϕ – CDF for $N(0,1)$

$\phi(-z) = 1 - \phi(z)$

$$\phi\left(\frac{0.003}{\sigma}\right) \ge \frac{1.99}{2} \iff \frac{0.003}{\sigma} \ge \phi^{-1}\left(\frac{1.99}{2}\right) = \text{qnorm}\left(\frac{1.99}{2}\right) = 2.58$$

$$\frac{0.003}{\sigma} \geq 2.58 \iff \sigma \leq \frac{0.003}{2.58} = \underline{0.0011}$$

sgr. 1

Exercise 69. Let X be a random variable whose distribution function is $F(x) = a + b \arctan \frac{x}{2}$, $x \in \mathbb{R}$. Find the constants a and b such that F satisfies the properties of a distribution function.

Exercise 70. The density function of a continuous random variable X is $f(x) = \frac{b}{a^2 + x^2}$, $x \in \mathbb{R}$. Find the coefficient b and the distribution function of X.

Find the PDF at ex 69 and the CDF at ex 70. Compute the E[X], V[X] and P(-1<=X<=1) for both exercises.

69. $F(x) = a + b \operatorname{arctg} \frac{x}{2}$, $x \in \mathbb{R}$

$$\begin{cases} F\text{ - nondecreasing} \iff b > 0 \\ F\text{ - continuous } \checkmark \\ \lim\limits_{x \to -\infty} F(x) = 0, \ \lim\limits_{x \to \infty} F(x) = 1 \end{cases}$$

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \left(a + b \operatorname{arctg} \frac{x}{2}\right) = a + b \cdot \frac{\pi}{2} = 1 \iff$$

$$\iff a + \frac{\pi}{2} b = 1$$

$$\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \left(a + b \operatorname{arctg} \frac{x}{2}\right) = a + b \cdot \left(-\frac{\pi}{2}\right) = 0 \iff a - \frac{\pi}{2} \cdot b = 0$$

$$\begin{cases} a - \frac{\pi}{2} b = 0 \\ a + \frac{\pi}{2} b = 1 \end{cases} \quad +$$

$$2a \quad / \quad = 1 \iff a = \frac{1}{2} \Rightarrow b = \frac{2}{\pi} a \iff b = \frac{1}{\pi}$$

f - odd $\iff f(-x) = -f(x)$

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \operatorname{arctg} \frac{x}{2}, (\forall)\, x \in \mathbb{R}$$

PDF: $f(x) = F'(x) = \frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{\pi} \cdot \frac{\cancel{4}^{\,2}}{x^2+4} \cdot \frac{1}{\cancel{2}} = \frac{2}{\pi(x^2+4)}$

$$f(x) = \frac{2}{\pi(x^2+4)}, (\forall)\, x \in \mathbb{R}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} \frac{2x}{\pi(x^2+4)}\, dx = \frac{1}{\pi} \ln(x^2+4) \Big|_{-\infty}^{\infty} = 0$$

$$\int_{-\infty}^{\infty} \frac{2x}{\pi(x^2+4)}\, dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{2x}{\pi(x^2+4)}\, dx = \lim_{a\to\infty} 0 = 0$$

$\left[\frac{1}{\pi} \ln(x^2+4) \Big|_{-a}^{a} = \frac{1}{\pi} \left(\ln(a^2+4) - \ln(a^2+4) \right) = 0 \right]$

$$V[X] = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \underbrace{E^2[X]}_{0} = \infty$$

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{2}{\pi(x^2+4)}\,dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{x^2+4-4}{x^2+4}\,dx = \frac{2}{\pi}\int_{-\infty}^{\infty} 1\,dx - \frac{8}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = \infty - \frac{\pi}{2} = \infty$$

$$\int_{-\infty}^{\infty} 1\,dx = \lim_{a\to\infty} \int_{-a}^{a} 1\,dx = \lim_{a\to\infty} x\Big|_{-a}^{a} = \lim_{a\to\infty} (\underbrace{a-(-a)}_{2a}) = \lim_{a\to\infty} 2a = \infty$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = \frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_{-\infty}^{\infty} = \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}\cdot\pi$$

$$P(-1 \le X \le 1) = \int_{-1}^{1} f(x)\,dx = F(1) - F(-1) = \cancel{\tfrac{1}{2}} + \frac{1}{\pi}\operatorname{arctg}\frac{1}{2} - \cancel{\tfrac{1}{2}} - \frac{1}{\pi}\operatorname{arctg}\left(-\frac{1}{2}\right) =$$

$$\operatorname{arctg}(-x) = -\operatorname{arctg}(x)$$

$$= \frac{2}{\pi}\operatorname{arctg}\frac{1}{2} = 0.295$$

70. $f(x)=\frac{b}{a^2+x^2}$, $(\forall)x\in\mathbb{R}$, $a\neq 0$

$$\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}=\frac{a^2}{a^2+x^2}\cdot\frac{1}{\cancel{a}}$$

$$\begin{cases} f(x)\geq 0 \iff b\geq 0 \\ \int_{-\infty}^{\infty} f(x)\,dx=1 \iff \int_{-\infty}^{\infty}\frac{b}{a^2+x^2}\,dx=\frac{b}{a}\operatorname{arctg}\frac{x}{a}\Big|_{-\infty}^{\infty}=\frac{b}{a}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{b}{a}\cdot\pi=1 \iff \boxed{b=\frac{a}{\pi}}\geq 0 \Rightarrow \underline{a>0} \\ f\text{-integrable} \end{cases}$$

f-continuous on $\mathbb{R}$ $\Rightarrow$ f-integrable

$$F(x)=\int_{-\infty}^{x}f(t)\,dt=\int_{-\infty}^{x}\frac{b}{a^2+t^2}\,dt=\frac{b}{a}\operatorname{arctg}\frac{t}{a}\Big|_{-\infty}^{x}=\frac{1}{\pi}\left(\operatorname{arctg}\frac{x}{a}-\left(-\frac{\pi}{2}\right)\right)=$$

$$=\frac{1}{\pi}\operatorname{arctg}\frac{x}{a}+\frac{1}{2}$$

$E[X]=\int_{-\infty}^{\infty}x f(x)\,dx=0$, $V[X]=\int_{-\infty}^{\infty}x^2 f(x)\,dx-E^2[X]=\dots$, $P(-1<X<1)=F(1)-F(-1)=\dots$

22. The distance between two points needs to be measured, in meters. The true distance between the points is 10 meters, but due to measurement error we can't measure the distance exactly. Instead, we will observe a value of $10+\epsilon$, where the error ϵ is distributed $\mathcal{N}(0, 0.04)$. Find the probability that the observed distance is within 0.4 meters of the true distance (10 meters). Give both an exact answer in terms of Φ and an approximate numerical answer.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\int e^{-x^2} dx$$

ε - r.v. which represents the measurement error

$\varepsilon \sim N(0, 0.04)$ $(N(\mu, \sigma^2))$

$\mu = 0, \sigma^2 = 0.04$ $(\sigma = 0.2)$

D - observed distance

$\underbrace{|D-10|} \leq 0.4 \Leftrightarrow -0.4 \leq D-10 \leq 0.4$

$\Leftrightarrow 9.6 \leq D \leq 10.4$

$P(D \in [9.6, 10.4]) = P(\varepsilon \in [-0.4, 0.4])$

$D = 10 + \varepsilon \Leftrightarrow D - 10 = \varepsilon$

$|\varepsilon| \leq 0.4 \Leftrightarrow -0.4 \leq \varepsilon \leq 0.4$

$D = 10 + \varepsilon \Rightarrow D \sim N(10, 0.04)$

$E[D] = E[10+\varepsilon] = E[10] + E[\varepsilon] = 10 + 0 = 10$

$V[D] = V[10+\varepsilon] = V[10] + V[\varepsilon] = 0 + 0.04 = 0.04$

$P(-0.4 \leq \varepsilon \leq 0.4) = \int_{-0.4}^{0.4} f(x)dx = F(0.4) - F(-0.4) = \text{pnorm}(0.4, 0, 0.2) - \text{pnorm}(-0.4, 0, 0.2) = 0.954$

dnorm(x, m, s) $\rightarrow$ PDF : $f(x)$

pnorm(z, m, s) $\rightarrow$ CDF : $F(z)$

qnorm(p, m, s) $\rightarrow$ $F^{-1}(p)$ (inverse CDF)

rnorm(n) $\rightarrow$ generates random values with the normal distribution

$\Phi \rightarrow$ CDF for $N(0,1)$ (standard normal distribution)

$$X \sim N(\mu, \sigma^2) \iff Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

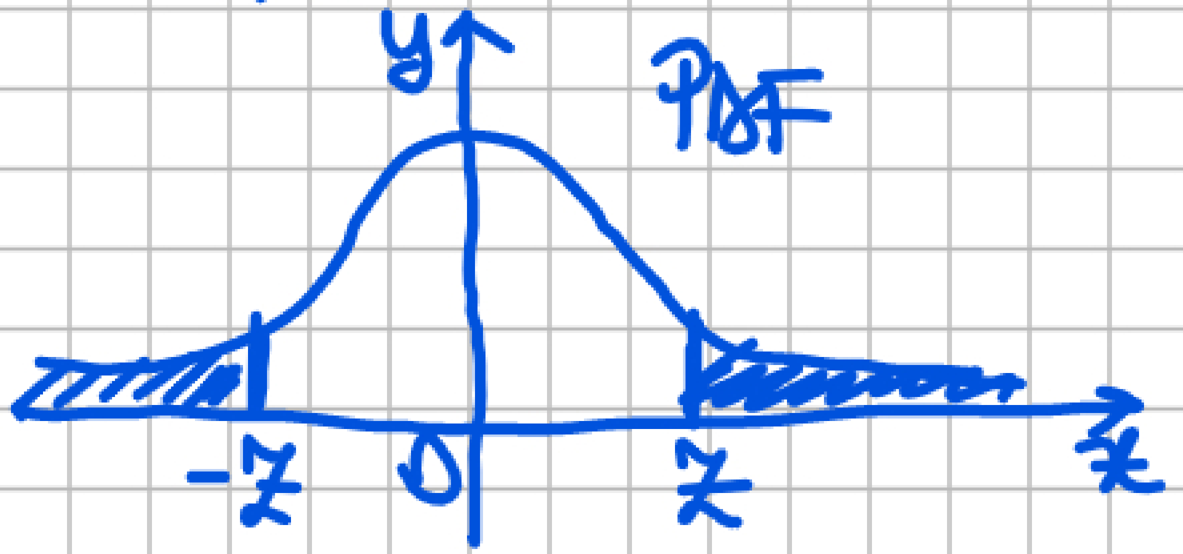
$$P(-0.4 \le \varepsilon \le 0.4) = \left(\frac{-0.4}{0.2} \le Z \le \frac{0.4}{0.2}\right) = P(-2 \le Z \le 2) = \Phi(2) - \Phi(-2) =$$

$$Z = \frac{\varepsilon}{\sigma}, \quad \Phi(-z) = 1 - \Phi(z)$$

$$= \Phi(2) - (1 - \Phi(2)) = \underline{2\Phi(2) - 1}$$

$$= 2\,\text{pnorm}(2) - 1$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt$$

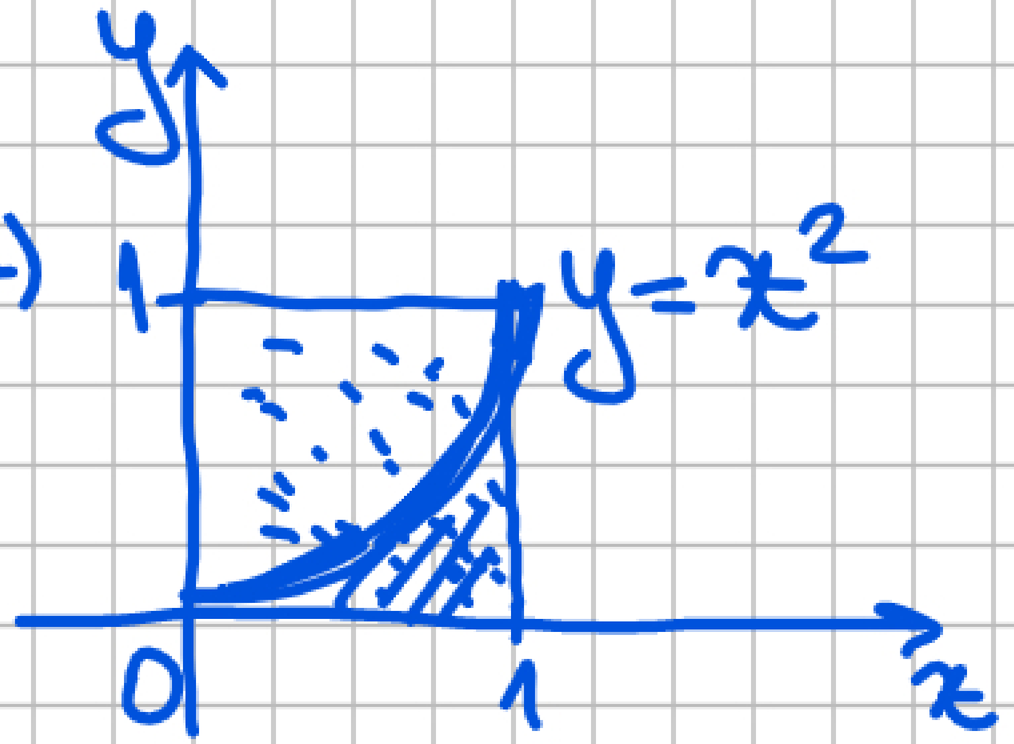


$$f(x)=\begin{cases}0, & x\le 0\\ ax, & x\in[0,1]\\ 0, & x\ge 1\end{cases}$$

$$F(x)=\int_{-\infty}^{x} f(t)\,dt=\begin{cases}0, & x\le 0\\ \int_0^x at\,dt, & x\in(0,1)\\ \int_0^1 f(t)\,dt, & x\ge 1\end{cases}$$

$$\int_0^1 f(t)\,dt = 1$$

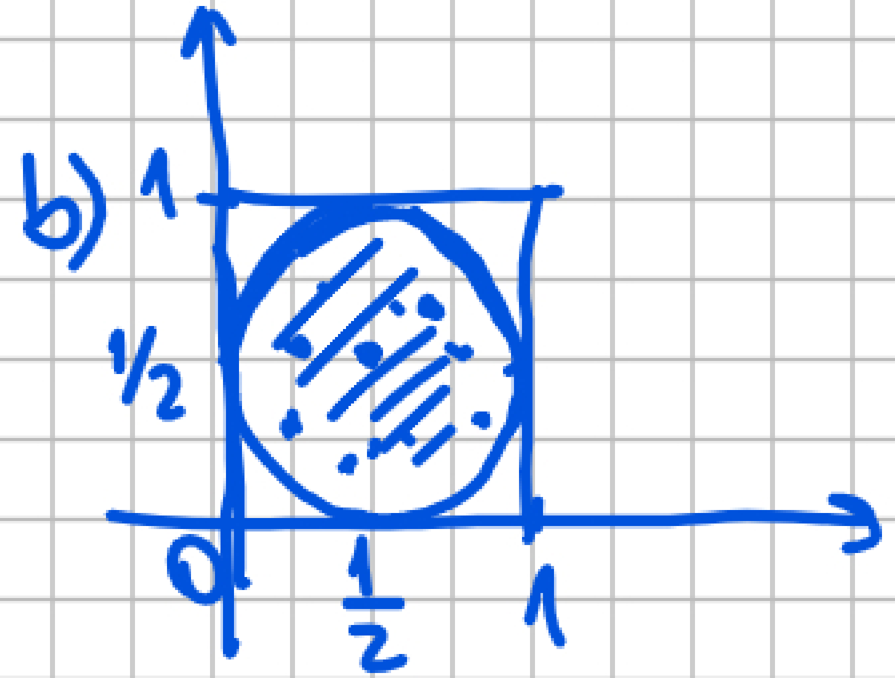
A1. a)



$(x,y) \longrightarrow y\le x^2$

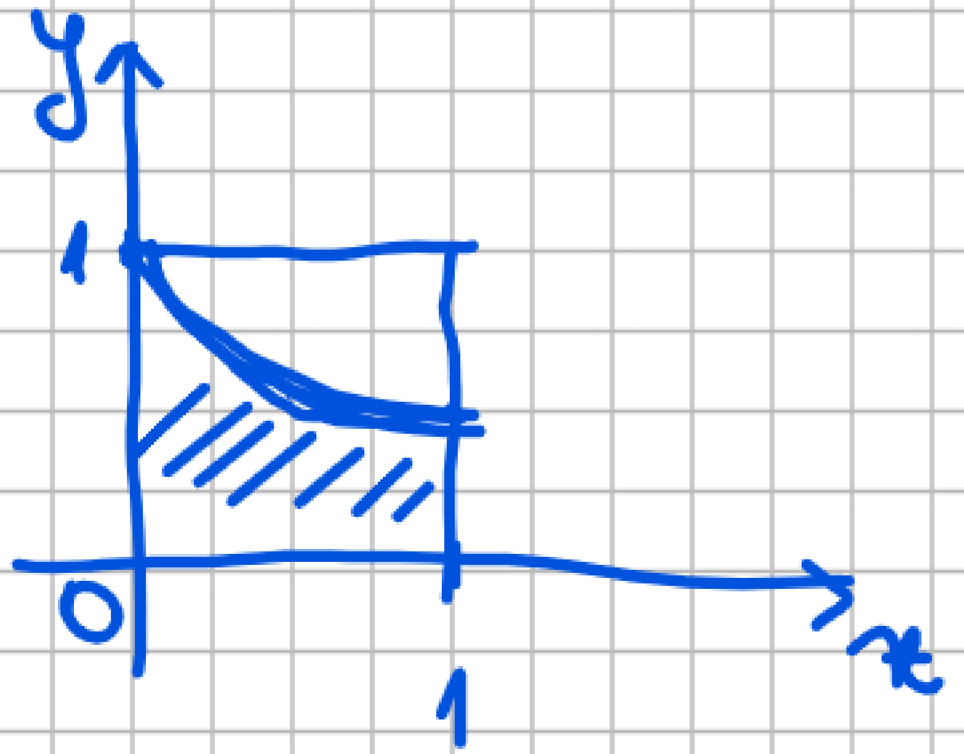
$$\int_0^1 f(x)\,dx=\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}=0.33$$

b)



$$(x-0{,}5)^2+(y-0{,}5)^2\le r^2=0{,}5^2$$

c)



$$y = \frac{1}{x+1}$$

$$y \le \frac{1}{x+1}$$

$$\int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \underline{\underline{\ln 2}}$$

Homework: assignments 4 and 5 from lab4.pdf

sgr. 4

8. The *Beta distribution* with parameters a — b has PDF

$$f(x) = a x^2(1-x), \text{ for } 0 < x < 1.$$

(We will discuss the Beta in detail in Chapter 8.) Let X have this distribution.

(a) Find the CDF of X.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

f is a valid PDF $\Leftrightarrow$ $f(x) \geq 0, (\forall) x \in \mathbb{R}$

f - integrable

$\int_{-\infty}^{\infty} f(x)dx = 1$

$f(x) = ax^2(1-x), \quad x \in (0,1)$

$f(x) = 0, (\forall) x \notin (0,1)$

$f(x) \geq 0 \Leftrightarrow a \geq 0$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} \underbrace{f(x)}_{=0}dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} \underbrace{f(x)}_{=0}dx = \int_{0}^{1} f(x)dx =$$

$$= \int_0^1 ax^2(1-x)dx = \int_0^1 ax^2dx - \int_0^1 ax^3dx =$$

$$= a\cdot\frac{x^3}{3}\Big|_0^1 - a\cdot\frac{x^4}{4}\Big|_0^1 = \frac{a}{3} - \frac{a}{4} = \frac{a}{12}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{a}{12} = 1 \iff \underline{\underline{a=12}}$$

a) CDF: $F:\mathbb{R}\to\mathbb{R}$, $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x\le 0 \\ 4x^3-3x^4, & x\in(0,1) \\ 1, & x\ge 1 \end{cases}$

$x\in(0,1)$, $$\int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{=0} + \int_{0}^{x} f(t)\,dt = \int_0^x 12t^2(1-t)\,dt = \int_0^x 12t^2\,dt - 12\int_0^x t^3\,dt$$

$$= \cancel{12}^{\,4}\cdot\frac{t^3}{\cancel{3}}\Big|_0^x - \cancel{12}^{\,3}\cdot\frac{t^4}{\cancel{4}}\Big|_0^x = 4x^3 - 3x^4$$

$x\ge 1$, $$\int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{=0} + \int_0^1 f(t)\,dt + \underbrace{\int_1^x f(t)\,dt}_{=0} = \int_0^1 f(t)\,dt = 1$$

d) $P\left(0<X<\frac{1}{2}\right)=\int_0^{\frac{1}{2}} f(t)\,dt = F\left(\frac{1}{2}\right)-F(0)=4\cdot\frac{1}{\cancel{2^3}_{2}}-3\cdot\frac{1}{2^4}=\frac{\cancel{8}\,1}{2}-\frac{3}{16}=\frac{5}{16}=0.313$ //

e) $E[X]=\int_{-\infty}^{\infty} x f(x)\,dx=\int_0^1 x f(x)\,dx=\int_0^1 x\cdot 12x^2(1-x)\,dx=12\int_0^1 x^3\,dx-12\int_0^1 x^4\,dx=$

$=\cancel{12}^{3}\cdot\frac{x^4}{\cancel{4}}\Big|_0^1-12\cdot\frac{x^5}{5}\Big|_0^1=\frac{\cancel{3}}{3}-\frac{12}{5}=\frac{3}{5}=0.6$

$V[X]=\int_{-\infty}^{\infty} x^2 f(x)\,dx-E^2[X]=\int_0^1 x^2 f(x)\,dx-0.36=\int_0^1 x^2\cdot 12x^2(1-x)\,dx-0.36=$

$=12\int_0^1 x^4\,dx-12\int_0^1 x^5\,dx-0.36=12\cdot\frac{x^5}{5}\Big|_0^1-\cancel{12}^{2}\cdot\frac{x^6}{\cancel{6}}\Big|_0^1-0.36=\frac{12}{5}-2-0.36=$

$=2.4-2.36=0.04$ //

Exercise 88. The capacitances of certain electronic components have a normal distribution with a mean $\mu = 174$ and a standard deviation $\sigma = 2.8$. If an engineer randomly selects a sample of size $n = 30$ components and measures their capacitances, what is the probability that the engineer's point estimate of the mean μ will be within the interval $(173, 175)$?

X - r.v. which represents the capacitances of an electronic component

$X \sim N(\mu, \sigma)$ $\mu = 174$, $\sigma = 2.8$, X_i - the capacitance of the i^{th} electronic component in the sample

$n = 30$

$X_i \sim N(\mu, \sigma)$, $i = \overline{1,n}$

$\bar{X}$ - sample mean $\Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} \rightarrow$ point estimate of the mean μ

$P(\bar{X} \in (173, 175)) = \int_{173}^{175} f(x)\,dx = F(175) - F(173) = \text{pnorm}(175, 174, 0.511) - \text{pnorm}(173, 174, 0.511)$

$= 0.95$

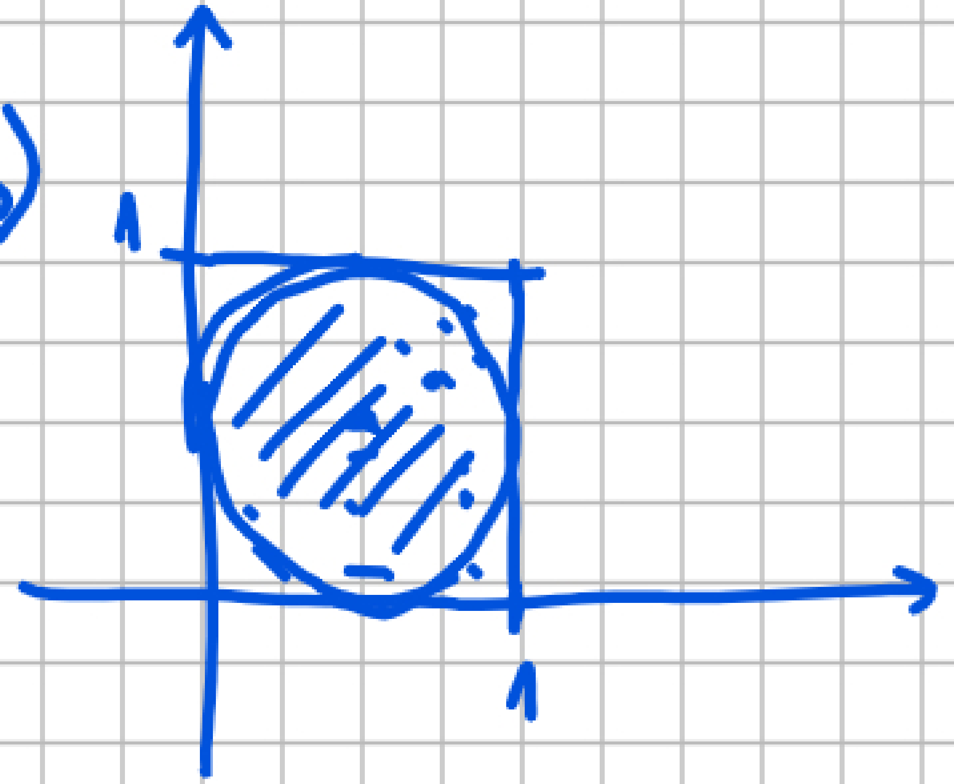
$F = \text{CDF} \rightarrow \text{pnorm}$; $\frac{\sigma}{\sqrt{n}} = \frac{2.8}{\sqrt{30}} = 0.511$

A1. a)

$(x,y) \longrightarrow y \le x^2$

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} = 0.33$$

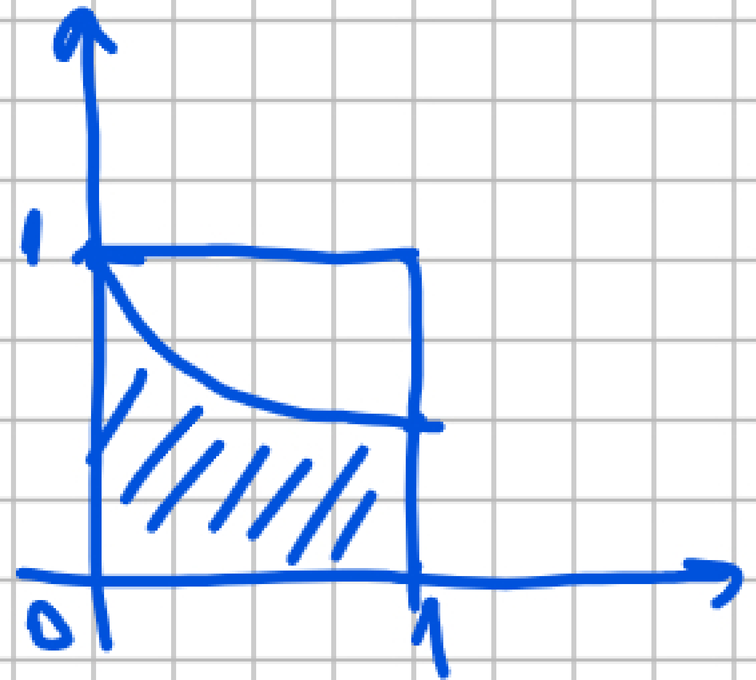
b)



$$\sqrt{(x-0.5)^2+(y-0.5)^2} \le r$$

$$A_{circle} = \pi \cdot r^2 \implies \pi = \frac{Area}{r^2}$$

c)



$$y \le \frac{1}{x+1}, \quad \int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln 2$$

A3. Write a function in R with param. $\underline{\lambda}$ and $\underline{n}$:

- generates n values in (0,1) $\rightarrow$ runif $\rightarrow$ u

- return $F^{-1}(u)$ $\left(-\log(1-u)/\lambda\right)$

Homework: assignment 2 and 4 from lab4.pdf