LAB 5. Simulating discrete random variables.

If we want to simulate a discrete random variable X having the probability mass function

$$P(X = x_i) = p_i \qquad \forall\ i = 1, 2, 3... \qquad \text{where } \sum_i p_i = 1$$

we consider a uniformly distributed random variable U over the interval $(0, 1)$, i.e. U is a random number generated in the interval $(0, 1)$. It can be easily seen that for any $a \in [0, 1]$ we have $P(U < a) = a$.
We set:

$$X = \begin{cases} x_1 & \text{if } U < p_1 \\ x_2 & \text{if } p_1 \le U < p_1 + p_2 \\ x_3 & \text{if } p_1 + p_2 \le U < p_1 + p_2 + p_3 \\ \vdots & \vdots \\ x_i & \text{if } \sum\limits_{j=1}^{i-1} p_j \le U < \sum\limits_{j=1}^{i} p_j \\ \vdots & \vdots \end{cases}$$

Since

$$P(X = x_i) = P\left(\sum_{j=1}^{i-1} p_j \le U < \sum_{j=1}^{i} p_j\right) = P\left(U < \sum_{j=1}^{i} p_j\right) - P\left(U < \sum_{j=1}^{i-1} p_j\right) = \sum_{j=1}^{i} p_j - \sum_{j=1}^{i-1} p_j = p_i$$

we can see that X has the desired distribution.

ASSIGNMENTS:

♣ **1.** Simulate a Bernoulli random variable with probability of success p.

♣ **2. Simulating a geometric random variable.** Simulate a discrete random variable with geometric distribution, with probability of success p, based on the method presented above.

$$P(X = i) = (1 - p)^{i-1} p \qquad \forall\ i = 1, 2, 3, ...$$

♣ **3. Simulating a binomial random variable.** A binomial random variable $X \sim B(n, p)$ can be easily simulated, since it can be expressed as the sum of n independent Bernoulli random variables. That is, if $U_1, U_2, ..., U_n$ are independent uniform random variables over the interval $(0, 1)$, then letting

$$X_i = \begin{cases} 1 & \text{if } U_i < p \\ 0 & \text{otherwise} \end{cases}$$

it follows that $X = \sum\limits_{i=1}^{n} X_i$ is a random variable with binomial distribution $B(n, p)$.

♣ **4. Simulating a Poisson random variable.** To simulate a Poisson random variable with mean λ, we generate independent uniform random variables $U_1, U_2, ..., U_i, ...$ over the interval $(0, 1)$, stopping at

$$N = \min\left\{ n \ : \ \prod_{i=1}^{n} U_i < e^{-\lambda} \right\}$$

The random variable $X = N - 1$ has the desired Poisson distribution.