Lab 5. Simulating discrete random variables

sgr. 3

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ p_1 & p_2 & p_3 & \dots \end{pmatrix}$$

$p_i \in [0,1]$, $\sum_{i=1}^{\infty} p_i = 1$.

$U \sim U(0,1) \rightarrow$ continuous r.v.

$\downarrow$
uniform distribution

To simulate (using a computer) a random variable $\equiv$ randomly generating a value of the random variable

- runif(1) $\longrightarrow U$
- if $U < p_1 \Rightarrow x_1$

  if $p_1 \le U < p_1 + p_2 \Rightarrow x_2$

  if $p_1 + p_2 \le U < p_1 + p_2 + p_3 \Rightarrow x_3$

  $\vdots$

(1) Simulate a Bernoulli random variable with probability p.

$X: \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

- write a function in R for this simulation
- call the function 10 times for a specific value for p.

Bernoulli $\longrightarrow$ function(p)

- generate a random nr. on [0,1] $\longrightarrow$ u = runif(1)
- if $u < 1-p \implies 0$
  else $\longrightarrow 1$

Bernoulli(0.3)
0.7

2) Geometric distribution

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & i & \dots \\ p & (1-p)p & \dots & & (1-p)^{i-1}\cdot p & \dots \end{pmatrix}$$

$p_1 + p_2 + \dots + p_{i-1}$, $i \geq 2$

• write a function in R with a parameter p

- u = runif(1)
- s = p
- i = 1
- $\boxed{\text{while}}$ (s ≤ u) { s = p_i + s
  i = i + 1 }
- return(i)

$u < p_1 \Rightarrow 1$

$p_1 \leq u < p_1 + p_2 \Rightarrow 2$

$p_1 + p_2 \leq u < p_1 + p_2 + p_3 \Rightarrow 3$

$\sum_{k=1}^{i-1} p_k \leq u < \sum_{k=1}^{i} p_k \Rightarrow i$

③ Binomial distribution : $X \sim Bin(n, p)$

$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & & & p_k & & \end{pmatrix}$ $\quad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

- write a function in R with param. n & p:
  - $u = runif(0,1)$
  - $s = (1-p)^n$
  - $i = 0$
  - while $(s \le u)$ $\{ s = s + p_i$
    $i = i+1 \}$
  - return(i)

- $X = X_1 + X_2 + \dots + X_n$
  $X_i \sim Bernoulli(p)$
- in a for call the Bernoulli function (a1) n times & sum the results
- return this sum.

$s = 0$
for$(\dots)$ $s = s + B(p)$
return s

Homework: Monty Hall problem

extra: assignments 1 and 2 from lab4.pdf (Monte Carlo method and Buffon needle)

sgr. 2

$$X: \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_k & \dots \\ p_1 & p_2 & p_3 & \dots & p_k & \dots \end{pmatrix}$$

$$\sum_{k=1}^{\infty} p_k = 1$$

Simulating the r.v. X means to randomly generate (i.e. printing on the screen / returning a value) a value of X.

U = random nr. in $[0,1]$ $\rightarrow$ runif( )

if $U < p_1 \Rightarrow x_1$

$p_1 \leq U < p_1 + p_2 \Rightarrow x_2$

$p_1 + p_2 \leq U < p_1 + p_2 + p_3 \Rightarrow x_3$

$\vdots$

$$\sum_{j=1}^{i-1} p_j \leq U < \sum_{j=1}^{i} p_j \Rightarrow \underline{\underline{x_i}}$$

1. Simulate a Bernoulli r.v. with probability p.

- write a function in R for this ; $p \to$ parameter of the function
- $u = runif(1)$

  if $u < 1-p \to$ return 0

  else $\to$ return 1

$$runif(\underset{\downarrow}{n}, min = \underset{\downarrow}{a}, max = \underset{\downarrow}{b})$$
$$\quad 1 \qquad\quad 0 \qquad\quad 1$$

$$X : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

3. Simulate a r.v. with the binomial distribution.

$X \sim Bin(n, p)$

$X = X_1 + X_2 + \dots + X_n$

$X_i \sim Bernoulli(p)$

- write a function with param. n & p
- call the function at A1 n times
- sum the results
- return the sum

② Simulate a r.v. with the geometric distribution

$$X : \begin{pmatrix} 1 & 2 & 3 \dots & k & \dots \\ p & (1-p)p & \dots & (1-p)^{k-1}\cdot p & \dots \end{pmatrix} \qquad P(X=k) = (1-p)^{k-1}\cdot p$$

- write a function in R with parameter p:

• $u = \text{runif}(0,1)$

if $u < p \Rightarrow 1$

if $p \le u < p + (1-p)\cdot p \Rightarrow 2$

if $p + (1-p)\cdot p \le u < p + (1-p)p + (1-p)^2 p \Rightarrow 3$

$\vdots$

if $S_{k-1} \le u < S_k \Rightarrow k$

$S_k = p + (1-p)\cdot p + \dots + (1-p)^{k-1}\cdot p =$

$= p \cdot \frac{1-(1-p)^k}{1-(1-p)} = \cancel{p} \cdot \frac{1-(1-p)^k}{\cancel{p}} = 1-(1-p)^k$

$S = p$
$k = 1$
while $u \geq S$

$S = S + (1-p)^{k-1}\cdot p$
$k = k+1$

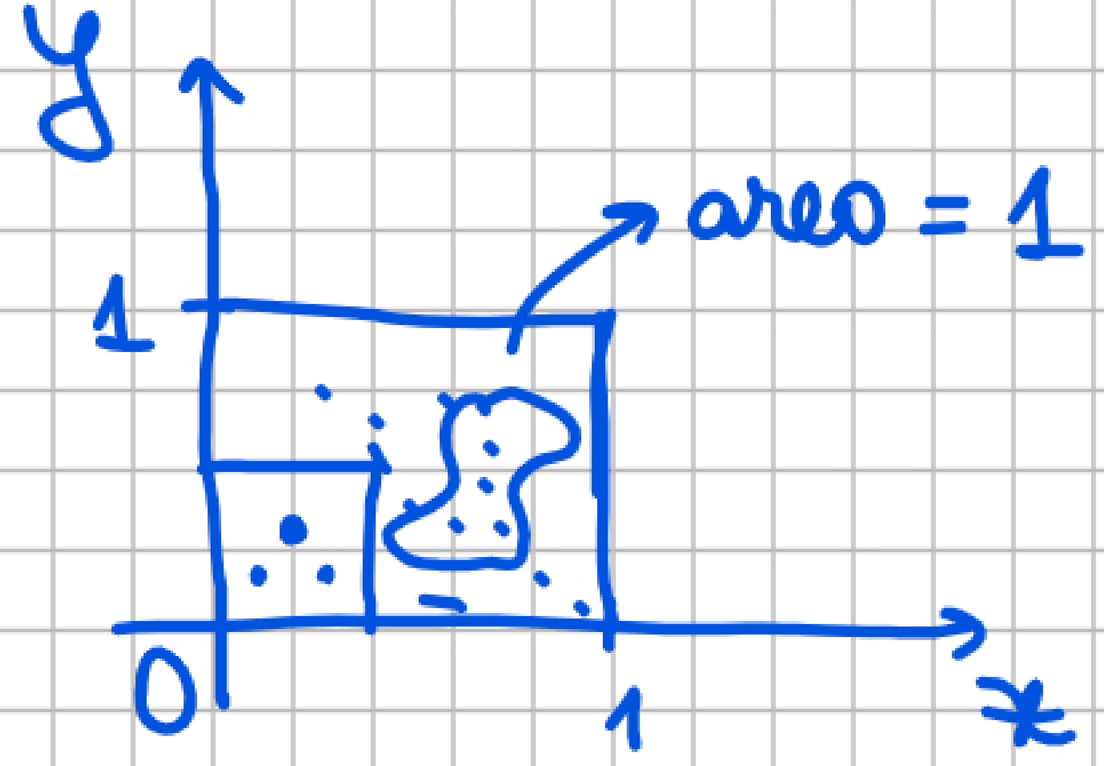
return(k)

④ Simulate a r.v. with the Poisson distribution.

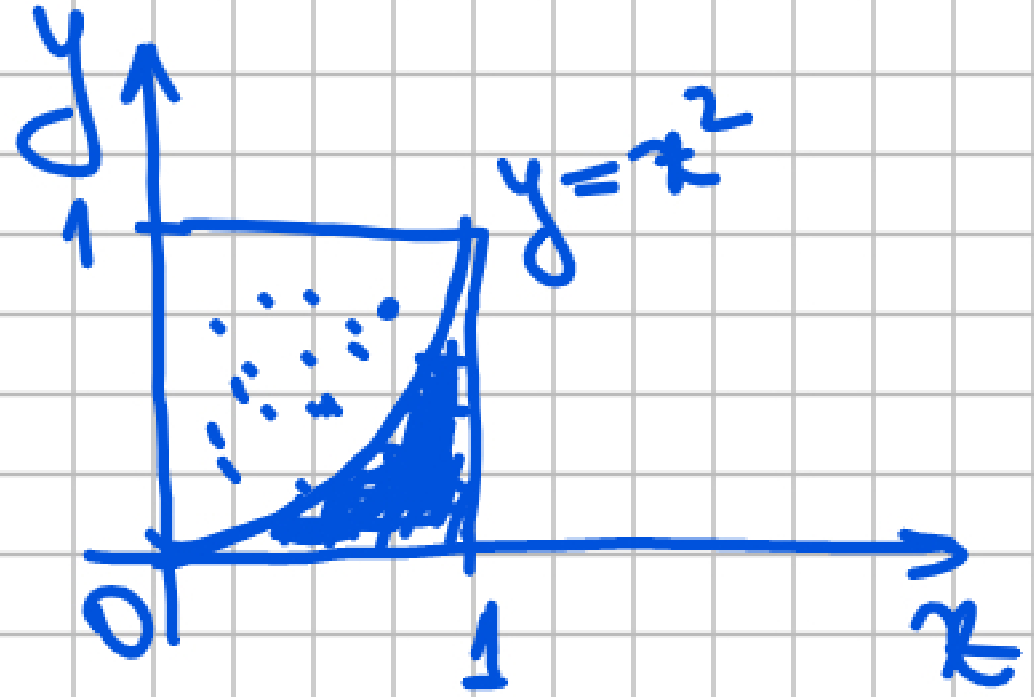
$$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & & \dots & & p_k & \dots \end{pmatrix} \quad P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

- write a function with parameter λ
  - generate random nr. in $(0,1)$ while their product $\geq \exp(-\lambda)$
  - return $(n-1)$

```
u = runif(1)
p = u ; n = 1
while (p >= exp(-λ))
  { u = runif(1)
    p = p * u
    n = n + 1
  }
return(n-1)
```



a)



- write a function in R:
  - choose at random many points inside the unit square (1000, 10 000, ...)

    $P(x,y)$ $\quad x,y \to$ random nr. in $(0,1)$
  - count nr. of points that fall under the curve $y=x^2$: $y \leq x^2$
  - return : nr. of points / nr. of generated points

H: A1 and 2 from lab4.pdf (Monty Carlo method and Buffon needle)

sgr. 5

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_k & \dots \\ p_1 & p_2 & p_3 & \dots & p_k & \dots \end{pmatrix}$$

$$\sum_{k=1}^{\infty} p_k = 1$$

Simulate the discrete r.v. X = randomly generating (selecting / printing / returning) a value of X taking in account the probabilities of these values.

- $U \sim Unif(0,1)$ $\to$ generate a random nr. in $(0,1)$
- if $\begin{cases} u < p_1 \Longrightarrow x_1 \\ p_1 \le u < p_1 + p_2 \Longrightarrow x_2 \\ p_1 + p_2 \le u < p_1 + p_2 + p_3 \Longrightarrow x_3 \\ \vdots \end{cases}$

$$\sum_{k=1}^{i-1} p_k \le u < \sum_{k=1}^{i} p_k \Longrightarrow x_i$$

① Simulate a Bernoulli r.v. with probability p.

$X \sim Bern(p)$ $\quad X : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

- write a function in R with parameter p:
  - selects/generate a random nr. in $(0,1)$ $\longrightarrow$ runif()
    $\downarrow u$
  - if $u < 1-p \Longrightarrow 0$
    else $\Longrightarrow 1$

③ Simulating a binomial random variable

$X \sim Bin(\underline{n, p})$

$$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & \dots & & \binom{n}{k} p^k (1-p)^{n-k} & \dots & \end{pmatrix} \qquad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$X = X_1 + X_2 + \dots + X_n$, X_i – independent Bernoulli r.v.s., $i = \overline{1,n}$

$$X_i : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

- write a function in R with parameters n & p
  - call the function defined at A1 $\underline{n}$ times $\longrightarrow$ for( )
  - save the results (in a variable)
  - return this sum

② Simulate a geometric r.v.

$X \sim Geom(p)$

$X: \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ p & (1-p)p & & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$ $P(X=k) = (1-p)^{k-1} \cdot p \,, \; k \geq 1$

Write a function in R with parameter p:

- $u \to$ random nr. in $(0,1)$ (runif)
- if $u < p \Rightarrow 1$

  $p \leq u < p + (1-p)p \Rightarrow 2$

  $p + (1-p)p \leq u < p + (1-p)p + \underline{(1-p)^2 p} \Rightarrow 3$

  $\vdots$

```
u = runif(1)
s = p ; k = 1
while (u>s)
{ s = s + (1-p)^k · p
  k = k + 1
}

return(k)
```

4. Simulating a Poisson r.v.

$X \sim Po(\lambda)$

$$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & & \dots & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \dots \end{pmatrix} \quad P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \; k \in \mathbb{N}$$

Write a function in R with parameter λ:

while
- generate randomly values in $(0,1)$ $\longrightarrow$ runif
- compute the product of these values
- stop when product $< e^{-\lambda}$

return (nr. of generated values $- 1$)

Homework: assignments 1 and 2 from lab4.pdf (Monte Carlo method and the Buffon needle)

sgr. 6

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ p_1 & p_2 & p_3 & \dots \end{pmatrix} \qquad \sum_{i=1}^{\infty} p_i = 1$$

- $U \sim Unif(0,1) \longrightarrow$ runif()
- if $U < p_1 \Rightarrow x_1$

$p_1 \leq U < p_1 + p_2 \Rightarrow x_2$

$p_1 + p_2 \leq U < p_1 + p_2 + p_3 \Rightarrow x_3$

$$\sum_{k=1}^{i-1} p_k \leq U < \sum_{k=1}^{i} p_k \Rightarrow x_i$$

① Simulate a Bernoulli r.v. with of success p.

$X \sim Bern(p)$

$$X : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

Write a function in R with parameter p:

- generate u a random nr. in $(0,1)$
- if $u < 1-p \Rightarrow 0$

else $\Rightarrow 1$

③ Simulating a binomial random variable.

$X \sim Bin(n,p)$

$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & \dots & & & \binom{n}{k}p^k(1-p)^{n-k} & \dots & \end{pmatrix}$ $\quad p_k = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

- generate u - a random nr. in $(0,1)$
- if $u < p_1 = (1-p)^n \Longrightarrow 0$

  $p_1 \le u < p_1 + p_2 \Longrightarrow 1$

  $\vdots$

  $p_1 + \dots + p_{k-1} \le u < p_1 + \dots + p_k \Rightarrow k$

$X = X_1 + X_2 + \dots + X_n$

$X_i \sim Bern(p)$, indep.

Write a function in R with param. n & p:

- generate a $u \in (0,1) \to$ runif
- call the function at ex. 1 $\underline{n}$ times $\to$ for
- sum the results in a variable
- return this sum

② Simulate a geometric random variable with probability of success p.

$X \sim Geom(p)$

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix} \qquad P(X=k) = (1-p)^{k-1} \cdot p, \ (\forall) k \geq 1$$

Write a function in R with parameter p:

- generate a random nr. in $(0,1) \longrightarrow u$ (runif)
- $S = p$; $k = 1$
- while (cond) $\xrightarrow{} u > S$ { $k = k+1$

  $S = S + (1-p)^{k-1} \cdot p$

  }

- return (k)

(4) Simulating a Poisson random variable with parameter λ.

Write a function in R with param. λ:

- generates u randomly in (0,1)
- $p = u$; $n = 1$
- while ($p \geq e^{-\lambda}$)

```
{ u = runif(1)
  p = p*u
  n = n+1
}
```

- return(n-1)

$X \sim Po(\lambda)$

$$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & & \dots & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \dots \end{pmatrix}$$

$$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Homework: assignments 1 and 2 from lab4.pdf

(Monte Carlo method and the Buffon needle)

sgr. 1

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ p_1 & p_2 & p_3 & \dots \end{pmatrix}$$

$$\sum_{k=1}^{\infty} p_k = 1$$

Simulate the r.v. X:

- generate a random nr. in $(0,1) \longrightarrow$ runif( ) ($\to u$)
- if $u < p_1 \Longrightarrow x_1$

$$p_1 \le u < p_1 + p_2 \Longrightarrow x_2$$

$$p_1 + p_2 \le u < p_1 + p_2 + p_3 \Longrightarrow x_3$$

$$\vdots$$

$$\sum_{k=1}^{i-1} p_k \le u < \sum_{k=1}^{i} p_k \Longrightarrow x_i$$

① Simulate a Bernoulli r.v. with probability p of success.

$X \sim Bern(p)$ $\qquad X : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

Write a function in R with parameter p:

- generate a random nr. in $(0,1)$ and store it in $\underline{\underline{u}}$
- if $u < 1-p \Rightarrow 0$
  else $\Rightarrow 1$

③ Simulate a binomial random variable with param. n & p.

$$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & \dots & & p_k & \dots & \end{pmatrix} \quad p_k = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = n \cdot p$$

$$X = X_1 + X_2 + \dots + X_n$$

X_i - independent Bernoulli r.v.s.

$X_i \sim Bern(p)$

Write a function in R with parameters n & p:

- in a for call the function at ex. 1
  . $\underline{n}$ times
- add the results & store in a variable
- return this sum

② Simulating a geometric random variable with parameter p.

$X \sim Geom(p)$

$$X: \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & p_k & \dots \end{pmatrix} \qquad p_k = P(X=k) = (1-p)^{k-1} \cdot p, \ (\forall)\, k \geq 1$$

$$E[X] = \frac{1}{p}$$

Write a function in R with param. p:

- generate a random nr. in $(0,1) \to u$
- if $u < p \Rightarrow 1$

$$p \leq u < p + (1-p)\cdot p \Rightarrow 2$$

$$\vdots$$

$$\sum_{k=1}^{i-1} (1-p)^{k-1} \cdot p \leq u < \sum_{k=1}^{i} (1-p)^{k-1} \cdot p \Rightarrow i$$

$\to$ <u>while</u>

```
s=p; k=1
while (u≥s) { k=k+1
              s=s+(1-p)^(k-1)·p
            }
return(k)
```

(4) Simulate a Poisson random variable with parameter λ.

$X \sim Po(\lambda)$, $X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & & \dots & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \dots \end{pmatrix}$ $P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $(\forall)\, k \in \mathbb{N}$

$E[X] = \lambda$

Write a function in R with parameter λ:

- p=runif(1); n=1
- while (p ≥ $e^{-\lambda}$)

```
{ p=p*runif(1)
  n=n+1
}
```

- return (n-1)

Homework: assignments 1 and 2 from lab4.pdf (Monte Carlo method and the Buffon needle)

sgr. 4

Hospital records show that of patients suffering from a certain disease, 75% die of it. What is the probability that of 10 randomly selected patients, 4 will recover? What is the probability that at most 2 patients die? What is the average number of patients that will die?

X - nr. of patients that die

$X \sim Bin(n, p)$

$n = 10$

$p = 75\% \ (0.75)$

$X: \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & \dots & & p_k & \dots & \end{pmatrix}$

$p_k = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$P(X=6) = \binom{10}{6} p^6 (1-p)^{10-6} = \binom{10}{6} \cdot 0.75^6 \cdot 0.25^4 = \underline{0.146}$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = \underbrace{\binom{10}{0}}_{1} p^0 (1-p)^{10} + \underbrace{\binom{10}{1}}_{10} p^1 (1-p)^9 + \underbrace{\binom{10}{2}}_{45} p^2 (1-p)^8 = 0.0004$

E[X] - average nr. of patients that are expected to die (out of the 10 selected patients)

$E[X] = n \cdot p = 10 \cdot 0.75 = 7.5$

- Binomial distribution :
  - n trials are performed and the nr. of successes is counted (we toss a coin $\underline{n}$ times & count the nr. of Heads)
- Geometric distribution :
  - trials are performed until we get the first succes (we toss a coin until we get the first Heads)
- Negative binomial distribution
  - trials are performed until we get $\underline{k}$ successes (we toss a coin until we get $\underline{5}$ Heads)
- Poisson distribution
  - nr. of (rare) events in a time unit (emails received in an hour)

2. Consider an urn with 5 blue balls and 7 yellow balls. You select balls one by one with replacement until you extract a blue ball. What is the probability that you need to extract 3 ball? What is the probability that you need to extract at least 2 balls? What is the average number of balls extracted until you find a blue one?

X - nr. of selections / extractions needed to find a blue ball

$X \sim Geom(p)$

$p = \frac{5}{12}$

$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ p & (1-p)p & \dots & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$ $\quad P(X=k) = (1-p)^{k-1} \cdot p$

$P(X=3) = (1-p)^2 \cdot p = \left(\frac{7}{12}\right)^2 \cdot \frac{5}{12} = \frac{49 \cdot 5}{12^3} = 0.142$

$1-p = 1 - \frac{5}{12} = \frac{7}{12}$

$P(X \geq 2) = 1 - P(X<2) = 1 - P(X=1) = 1-p = \frac{7}{12} = 0.583$

$E[X] = \frac{1}{p} = \frac{12}{5} = 2.4$

3. A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. Let p be the probability that a randomly selected couple agrees to participate. If p = 0.15, what is the probability that 15 couples must be asked before 5 are found who agree to participate?

X – nr. of couples asked to participate until we find 5 couples that agree

$X \sim NB(k, p)$

$k = 5$

$p = 0.15$

$$X : \begin{pmatrix} \overset{k}{5} & \overset{k+1}{6} & 7 & \dots & s & \dots \end{pmatrix}$$

$$P(X=s) = \binom{s-1}{k-1} \cdot p^k (1-p)^{s-k}$$

$$\underbrace{\underset{1}{1} \quad \underset{2}{0} \quad \underset{3}{1} \quad \dots \quad \underset{s-1}{1}}_{k-1 \text{ successes}} \quad \underset{s}{1}$$

$$P(X=15) = \binom{15-1}{5-1} \cdot p^5 (1-p)^{10} = \binom{14}{4} \cdot 0.15^5 \cdot 0.85^{10}$$
$$= 0.015$$

what is the probability that at least 10 couples needed to be asked?

$$P(X \geq 10) = 1 - P(X < 10) = 1 - [P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)]$$
$$= 1 - \left[\binom{4}{4} \cdot p^5 (1-p)^0 + \binom{5}{4} p^5 (1-p) + \binom{6}{4} p^5 (1-p)^2 + \binom{7}{4} p^5 (1-p)^3 + \binom{8}{4} p^5 (1-p)^4\right] = \dots$$

4. At a call-center the average number of telephone calls in an hour is 15. What is the probability that in the next hour 7 phone calls are answered? What is the probability that there are at most 5 in coming phone calls?

X – nr. of incoming phone calls in an hour

$X \sim Po(\lambda)$

$\lambda = 15$

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \dots \end{pmatrix}$$

$$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$P(X=7) = e^{-15} \cdot \frac{15^7}{7!} = 0.010$$

$$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) =$$

$$= e^{-\lambda} + e^{-\lambda} \cdot \lambda + e^{-\lambda} \cdot \frac{\lambda^2}{2} + e^{-\lambda} \cdot \frac{\lambda^3}{6} + e^{-\lambda} \cdot \frac{\lambda^4}{24} + e^{-\lambda} \cdot \frac{\lambda^5}{120}$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24} + \frac{\lambda^5}{120}\right) = 0.003$$

Simulating discrete random variables

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ p_1 & p_2 & p_3 & \cdots \end{pmatrix} \qquad \sum_{i=1}^{\infty} p_i = 1$$

$U \sim Unif(0,1) \longrightarrow$ runif()

- if $u < p_1 \Longrightarrow x_1$

$p_1 \leq u < p_1 + p_2 \Longrightarrow x_2$

$p_1 + p_2 \leq u < p_1 + p_2 + p_3 \Longrightarrow x_3$

$\vdots$

$$\sum_{i=1}^{k-1} p_i \leq u < \sum_{i=1}^{k} p_i \Longrightarrow x_k$$

① Simulate a Bernoulli r.v. with probability of success p.

$X \sim Bern(p)$

$X: \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

Write a function in R with parameter p:

- randomly generates a value in $(0,1) \rightarrow u$
  (runif())

- if $u < 1-p \implies 0$
  else $\implies 1$

② Simulating a binomial r.v.

$X \sim Bin(n, p)$

$X = X_1 + X_2 + \ldots + X_n$

X_i - independent. Bernoulli r.v.s.

$X_i \sim Bern(p)$

Write a function in R with parameters n & p:

- calls the function at ex. 1 $\underline{n}$ times $\longrightarrow$ for loop
- sums the results & stores the sum in a variable
- returns this sum

Homework: A2 and 4 from lab5.pdf and A1 and 2 from lab4.pdf (the Monte Carlo method and Buffon's needle)