Calculus - Lecture 9

Differentiability of functions of several real variables.

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Partial derivatives - example

Heat index I depends on temperature T and relative humidity H: $I = f(T, H)$.

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Fixing $H = 70\%$, consider $g(T) = f(T, 70)$ - describes how the heat index I increases as the temperature T increases when the relative humidity is 70%.
Rate of change:

$$g'(96) = \lim_{h\to 0} \frac{g(96+h) - g(96)}{h} = \lim_{h\to 0} \frac{f(96+h, 70) - f(96, 70)}{h} = \frac{\partial f}{\partial T}(96, 70)$$

Partial derivatives - example

For $h = 2$ we obtain: $g'(96) \simeq \dfrac{f(98,70) - f(96,70)}{2} = \dfrac{133 - 125}{2} = 4.$

For $h = -2$ we obtain: $g'(96) \simeq \dfrac{f(94,70) - f(96,70)}{-2} = \dfrac{118 - 125}{-2} = 3.5.$

Therefore, taking the mean value, we obtain the following approximation:

$$\frac{\partial f}{\partial T}(96,70) = g'(96) \simeq 3.75$$

$\Longrightarrow$ When the actual temperature is $96°F$ and the relative humidity is 70%, the heat index rises by about $3.75°F$ for every degree that the actual temperature rises.

Similarly,

$$\frac{\partial f}{\partial H}(96,70) \simeq 0.9$$

$\Longrightarrow$ When the actual temperature is $96°F$ and the relative humidity is 70%, the heat index rises by about $0.9°F$ for every percent that the relative humidity rises.

Partial derivatives - definition

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued function of n variables and $a = (a_1, a_2, ..., a_n) \in \text{Int}(A)$.

The function f is said to be partially differentiable with respect to x_i at a if the following limit exists and is finite

$$\lim_{h\to 0} \frac{f(a_1, ..., a_{i-1}, a_i + h, a_{i+1}, ..., a_n) - f(a_1, ..., a_{i-1}, a_i, a_{i+1}, ..., a_n)}{h}$$

The value of this limit is denoted by $\frac{\partial f}{\partial x_i}(a)$ and is called the partial derivative of f with respect to x_i at a.

The vector

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), ..., \frac{\partial f}{\partial x_n}(a)\right)$$

is called gradient vector of f at a.

Remarks

- To calculate partial derivatives, one has to differentiate (in the normal manner) with respect to x_i keeping all the other variables fixed.
- All obvious rules for partially differentiating sums, products and quotients can be used.
- The partial differentiability of a vector valued function of n real variables is equivalent to the partial differentiability of all the scalar components.
- For a function of two variables $f(x,y)$ the partial derivatives are:

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Example

Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is continuous, as:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\in[0,1]} = 0 = f(0,0).$$

The partial derivatives at $(0,0)$ are computed as:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Example

The partial derivatives at $(x,y) \neq (0,0)$ are computed using the usual derivative formulas:

$$\frac{\partial f}{\partial x} = f_x(x,y) = \left(\frac{x^3}{x^2+y^2}\right)'_x = \frac{3x^2(x^2+y^2) - 2x \cdot x^3}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \left(\frac{x^3}{x^2+y^2}\right)'_y = \frac{-2x^3y}{(x^2+y^2)^2}$$

Therefore, the partial derivatives are:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{x^4+3x^2y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{-2x^3y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Interpretation of partial derivatives

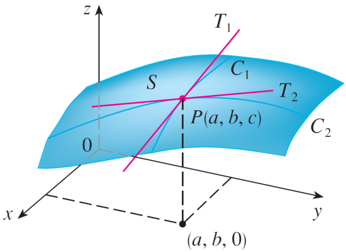
Rates of change: If $z = f(x, y)$ then:

- $\dfrac{\partial f}{\partial x}$ represents the rate of change of z with respect to x when y is fixed;
- $\dfrac{\partial f}{\partial y}$ represents the rate of change of z with respect to y when x is fixed.

Slopes: $z = f(x, y)$ is a surface S in $\mathbb{R}^3$.

$P(a, b, c)$ a point on this surface: $c = f(a, b)$.

$\dfrac{\partial f}{\partial x}(a, b) = f_x(a, b)$ and $\dfrac{\partial f}{\partial y}(a, b) = f_y(a, b)$ represent slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of the surface S in the planes $y = b$ and $x = a$.



Partial derivatives - example

The body mass index of a person is defined as

$$B(m,h) = \frac{m}{h^2}, \quad \text{where } m \text{ and } h \text{ are the mass and height of a person.}$$

$$\frac{\partial B}{\partial m} = \frac{1}{h^2} \quad \Longrightarrow \quad \frac{\partial B}{\partial m}(64, 1.68) = \frac{1}{(1.68)^2} \simeq 0.35 \ (kg/m^2)/kg$$

This is the rate at which the person's BMI increases with respect to his weight when he weighs $64\ kg$ and his height is $1.68\ m$. If his weight increases by $1\ kg$, and his height remains unchanged, then his BMI will increase by about 0.35.

$$\frac{\partial B}{\partial h} = \frac{-2m}{h^3} \quad \Longrightarrow \quad \frac{\partial B}{\partial h}(64, 1.68) = -\frac{128}{(1.68)^3} \simeq -27 \ (kg/m^2)/m^2$$

This is the rate at which the person's BMI increases with respect to his height when he weighs $64\ kg$ and his height is $1.68\ m$. If the man is still growing and his weight stays unchanged while his height increases by a small amount, say $1\ cm$, then his BMI will decrease by about $27(0.01) = 0.27$.

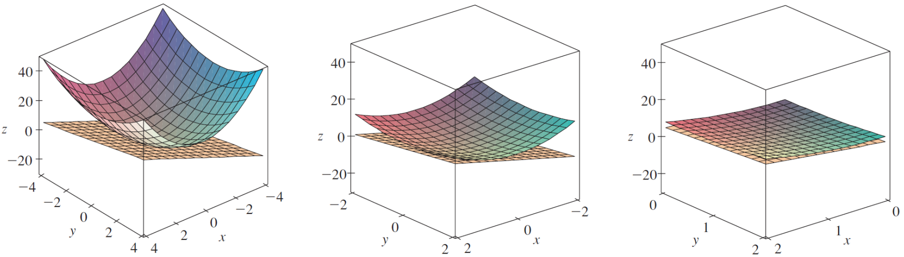
Tangent planes and partial derivatives

If the function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example. The equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $P(1, 1, 3)$ is:

$$z - 3 = 4(x - 1) + 2(y - 1) \qquad \Longrightarrow \qquad z = 4x + 2y - 3$$



Directional derivatives

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued function of n variables and $a \in \mathrm{Int}(A)$ and $u \in \mathbb{R}^n$ s.t. $\|u\| = 1$.

If the following limit exists and is a finite real number

$$\lim_{h\to 0} \frac{f(a+h\cdot u)-f(a)}{h} = \lim_{h\to 0} \frac{f(a_1+hu_1, a_2+hu_2, ..., a_n+hu_n)-f(a_1, a_2, ..., a_n)}{h}$$

it is called the directional derivative in the direction u of f at the point a and it is denoted by $\nabla_u f(a)$.

! Partial derivatives are special cases of directional derivatives:
The directional derivative of f at a in the direction $e_i = (0, ..., 0, 1, 0, ..., 0)$ is

$$\nabla_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a) \qquad i = \overline{1, n}$$

! Relationship between directional derivative and gradient:

$$\nabla_u f(a) = \nabla f(a) \cdot u \qquad (\text{where } \|u\| = 1)$$

Directional derivative - example

Example. Compute the directional derivative of the function

$f(x, y, z) = x^2 + xy + z^2$ in the direction $u = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ at the point $a = (1, 2, 3)$

First, we compute the gradient vector:

$$\nabla f = (2x + y, x, 2z) \quad \Longrightarrow \quad \nabla f(a) = (4, 1, 6)$$

Therefore, the directional derivative is:

$$\nabla_u f(a) = \nabla f(a) \cdot u = (4, 1, 6) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = 4 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} + 6 \cdot \frac{2}{3} = 6.$$

Directional derivative - example

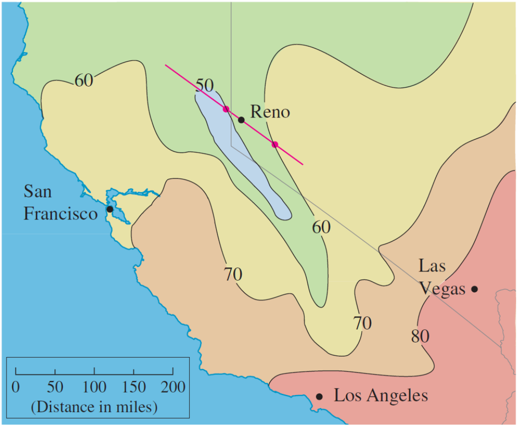
Example. Temperature map.

The unit vector directed toward the southeast:

$$u = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

The rate of change of temperature when we travel southeast:

$$\nabla_u T \simeq \frac{60 - 50}{75} \simeq 0.13^\circ \text{F/mi}$$



Differentiability

Theorem

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued function of n variables, and $a \in Int(A)$.

If the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = \overline{1, n}$ exist in a neighborhood of a and they are continuous at a, then the following equality holds:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0$$

Consequence: In a neighborhood of the point a (i.e. when $\|h\|$ is small), the following linear approximation holds:

$$f(a+h) \simeq f(a) + \nabla f(a) \cdot h \quad \text{(for } \|h\| \text{ small)}$$

Differentiability - definition

A real valued function of n variables $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ is said to be differentiable at a if the following conditions hold:

- it is partially differentiable at a with respect to every variable x_i
- $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0.$

The Fréchet derivative of f at a: the function $d_a f : \mathbb{R}^n \to \mathbb{R}^1$ defined by

$$d_a f(h) = \nabla f(a) \cdot h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot h_i$$

- The Fréchet derivative $d_a f : \mathbb{R}^n \to \mathbb{R}^1$ is a linear function on $\mathbb{R}^n$ (it is a first degree polynomial of $h_1, h_2, ..., h_n$).
- For $\|h\| = 1$, we have $d_a f(h) = \nabla_h f(a)$.
- If $f : \subset \mathbb{R}^n \to \mathbb{R}^1$ is differentiable at $a \in A$, then it is continuous at a.

Differentiability - definition

A vector valued function of n variables $f = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \text{Int}(A)$ if every scalar component f_j, $j = \overline{1, m}$ of f is differentiable at a.

The Fréchet derivative of f at a is the function $d_a f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$d_a f(h) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(a) \cdot h_i \right) \cdot e_j \qquad \text{where } e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m.$$

The matrix of the linear function $d_a f$ is called the Jacobi matrix of f at a:

$$J_a(f) = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{m \times n}$$

We have $d_a f(h) = J_a(f) \cdot h$.

Differentiability - examples

Example 1. For the real valued function $f(x,y,z) = x^2 + xy + z^2$, the Fréchet derivative of at the point $a = (1,2,3)$ is the function $d_a f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$d_a f(h) = \nabla f(1,2,3) \cdot (h_1, h_2, h_3) = (4,1,6) \cdot (h_1, h_2, h_3) = 4h_1 + h_2 + 6h_3$$

Example 2. For the vector valued function $f(x,y,z) = (x^2 + z^2, xy)$, the Jacobi matrix is:

$$J(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 0 & 2z \\ y & x & 0 \end{pmatrix}$$

and hence, the Jacobi matrix at the point $a = (1,2,3)$ is

$$J_a(f) = \begin{pmatrix} 2 & 0 & 6 \\ 2 & 1 & 0 \end{pmatrix}$$

The Fréchet derivative of is the function $d_a f : \mathbb{R}^3 \to \mathbb{R}^2$ given by:

$$d_a f(h) = J_a(f) \cdot h = \begin{pmatrix} 2h_1 + 6h_3 \\ 2h_1 + h_2 \end{pmatrix}$$

Differentiability - examples

Example 3. For the function

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

we have see that the partial derivatives at the point $(0,0)$ are

$$\frac{\partial f}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0 \quad \Longrightarrow \quad \nabla f(0,0) = (1,0).$$

Based on the definition, we check if the function is differentiable at $(0,0)$ by computing the limit:

$$L = \lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\|(h_1,h_2)\|} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3}{h_1^2+h_2^2} - 0 - (1,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3}{h_1^2+h_2^2} - h_1}{\sqrt{h_1^2+h_2^2}}$$

Differentiability - examples

Hence, simplifying the previous expression leads to:

$$L = \lim_{(h_1,h_2)\to(0,0)} \underbrace{\frac{-h_1 h_2^2}{(h_1^2+h_2^2)^{3/2}}}_{g(h_1,h_2)}$$

We observe that in the above limit, we have the ratio of two expressions of equal order 3, and we show that the limit does not exist:

along the horizontal axis: $g(h_1,0) = 0 \overset{h_1\to 0}{\longrightarrow} 0$

along the first bisector: $g(h_1,h_1) = \dfrac{-h_1^3}{(2h_1^2)^{3/2}} = -\dfrac{1}{2\sqrt{2}} \overset{h_1\to 0}{\longrightarrow} -\dfrac{1}{2\sqrt{2}}$

Conclusion: The function f is not differentiable at $(0,0)$.

Properties

Composite rule.

Let $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}^p$.

If f is differentiable at $a \in \text{Int}(A)$ and g is differentiable at $f(a) = b \in \text{Int}(B)$, then $h = g \circ f$ is differentiable at a and

$$d_a h = d_b g \circ d_a f$$

The Jacobi matrix of h at a is the product of the Jacobi matrix of g at b and the Jacobi matrix of f at a:

$$J_a(g \circ f) = J_b(g) J_a(f).$$

Inverse rule.

Let $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ be a bijection where A, B are open subsets of $\mathbb{R}^n$.

If f is differentiable at $a \in A$ and f^{-1} is differentiable at $b = f(a)$, then $d_a f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and

$$(d_a f)^{-1} = d_{f(a)} f^{-1}$$