# Calculus - Lecture 7

## Fourier series.

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# FOURIER SERIES - Introduction

The Fourier series of a piecewise continuous function $f$ defined on the interval $[-\pi, \pi]$ is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_2 \sin 3x + \dots$$

where the Fourier coefficients $a_n$, $b_n$ are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{for } n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad \text{for } n = 1, 2, \dots$$

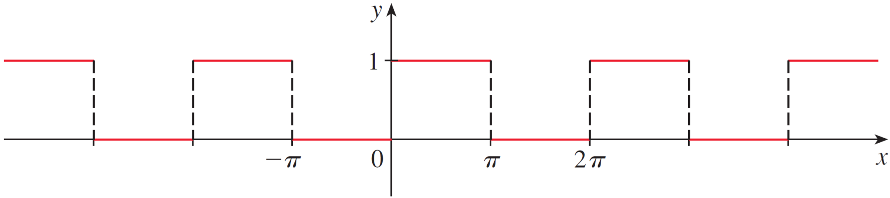
# FOURIER SERIES - Introduction

- Joseph Fourier (1768-1830) $\rightarrow$ heat conduction
- Daniel Bernoulli and Leonard Euler $\rightarrow$ vibrating strings and astronomy
- Expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series.
- Astronomical phenomena, heartbeats, tides, vibrating strings, etc.. are usually periodic, so it makes sense to express them in terms of periodic functions.

# Example 1: square-wave function

Consider the square-wave function

$$f(x) = \begin{cases} 0 & , \ x \in [-\pi, 0) \\ 1 & , \ x \in [0, \pi] \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x), \ \forall x \in \mathbb{R}.$$




REMINDER:

$$\cos n\pi = (-1)^n$$

$$\sin n\pi = 0$$

# Example 1: square-wave function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi} \cos nx dx = \frac{1}{\pi}\left.\frac{\sin nx}{n}\right|_0^{\pi} = 0 \quad , \forall n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi} \sin nx dx = \frac{1}{\pi}\left.\frac{-\cos nx}{n}\right|_0^{\pi} = \begin{cases} 0 & \text{, if } n \text{ - even} \\ \frac{2}{n\pi} & \text{, if } n \text{ - odd} \end{cases}$$

Therefore, the Fourier series is:

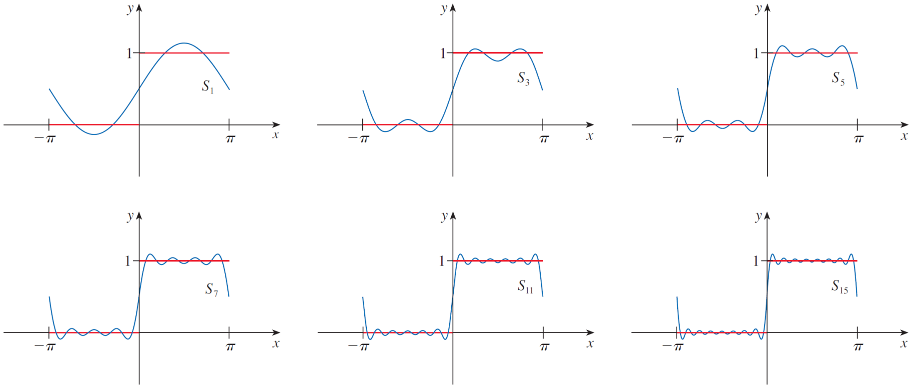
$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin(2k+1)x.$$

For example, the partial sum of order $7$ is:

$$S_7(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \frac{2}{7\pi}\sin 7x.$$

# Example 1: square-wave function

Different partial sums $S_n$ of the Fourier series are shown below:




As $n$ increases, the partial sum $S_n(x)$ becomes a better approximation of the function $f(x)$ (except at the points where $f$ is discontinuous).

# Fourier Convergence Theorem

## Theorem

*Let $f$ be a piecewise continuous function defined on the interval $[-\pi, \pi]$ and extended by periodicity outside it.*

*We denote by $S_n(x)$ the $n$-th partial sum of the Fourier series defined above.*

*If $f(x)$ has finite left- and right-hand side derivatives at its points of discontinuity, then:*

- *if $x = x_0$ is a point where $f$ is continuous, then*

$$\lim_{n\to\infty} S_n(x_0) = f(x_0).$$

- *if $x = x_0$ is a point of discontinuity of $f$, then*

$$\lim_{n\to\infty} S_n(x_0) = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right].$$

# Change of the origin of the fundamental interval

If $f$ is a piecewise continuous function defined on the fundamental interval $[\alpha - \pi, \alpha + \pi]$ and by periodic extension outside of it, then the Fourier coefficients are:

$$a_n = \frac{1}{\pi} \int_{\alpha-\pi}^{\alpha+\pi} f(x) \cos nx \, dx \quad \text{for } n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{\alpha-\pi}^{\alpha+\pi} f(x) \sin nx \, dx \quad \text{for } n = 1, 2, \dots$$

The Fourier series converges at every point where $f$ is continuous and:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

# Change of the interval length

If $f$ is a piecewise continuous function defined on the interval $[-L, L]$ and by periodic extension outside of it, then the Fourier coefficients are given by

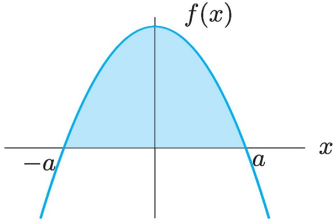
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for } n = 0, 1, 2, \ldots$$

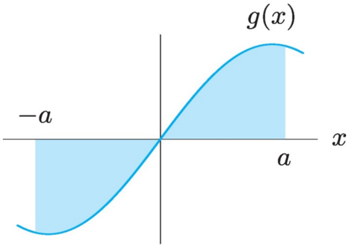
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for } n = 1, 2, \ldots$$

The Fourier series converges at every point where $f$ is continuous and:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

# Odd and even functions




If $f(x)$ is even: $f(-x) = f(x)$, we have:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If $f(x)$ is odd: $f(-x) = -f(x)$, we have:

$$\int_{-a}^{a} f(x)dx = 0$$

# Fourier series of even functions

Consider $f : [-\pi, \pi] \to \mathbb{R}$, extended by periodicity to $\mathbb{R}$.

If the function $f$ is even, then:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{\text{even}}\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd}}\,dx = 0$$

Therefore, the Fouries series is a cosine series:

$$\frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + a_n\cos nx + \ldots$$

If the main interval is $[-L, L]$, the results are adapted accordingly!

# Fourier series of odd functions

Consider $f : [-\pi, \pi] \to \mathbb{R}$, extended by periodicity to $\mathbb{R}$.

If the function $f$ is odd, then:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cos nx}_{\text{odd}} \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \sin nx}_{\text{even}} \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx$$

Therefore, the Fouries series is a sine series:

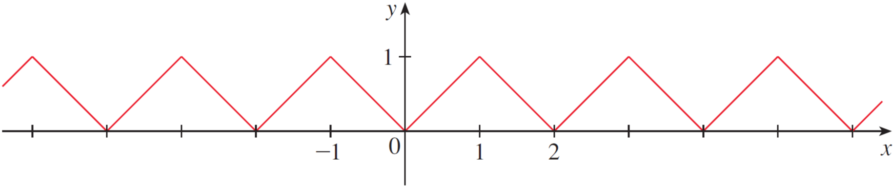
$$b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots$$

If the main interval is $[-L, L]$, the results are adapted accordingly!

# Example 2: triangle-wave function

Find the Fourier series of the function

$$f(x) = |x|,\ x \in [-1, 1] \quad \text{and} \quad f(x+2) = f(x),\ x \in \mathbb{R}.$$




The function is even: $|-x| = |x|$. Therefore: $b_n = 0$.

$$a_0 = \int_{-1}^{1} f(x)dx = 2\int_0^1 x dx = x^2|_0^1 = 1.$$

# Example 2: triangle-wave function

For $n \geq 1$, we have:

$$a_n = 2\int_0^1 f(x)\cos n\pi x dx = 2\int_0^1 x\cos n\pi x dx =$$

$$= 2\left( x\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x dx \right) = \frac{2}{n^2\pi^2}\cos n\pi x|_0^1 = \frac{2((-1)^n - 1)}{n^2\pi^2}$$

Therefore:

$$a_n = \begin{cases} 0 & \text{, if n is even} \\ -\dfrac{4}{n^2\pi^2} & \text{, if n is odd} \end{cases}$$

In conclusion, the Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}\cos((2k+1)\pi x) \quad \forall\, x \in \mathbb{R}.$$

# Example 2: triangle-wave function

This Fourier series helps us evaluate the sum of the series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Indeed, replacing $x = 1$ in the Fourier series, we obtain:

$$f(1) = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos((2k+1)\pi) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}$$

As $f(1) = 1$, we obtain:

$$\sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} = \frac{1}{2}$$

and therefore:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

# Parseval's identity

For a piecewise continuous function $f : [-L, L] \to \infty$, extended by periodicity to $\mathbb{R}$, with the Fourier coefficients $a_n$, $b_n$, Parseval's identity holds:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$$

Example 2. By Parseval's identity, we obtain:

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{16}{(2k+1)^4\pi^4} = \int_{-1}^{1} x^2 dx$$

and therefore:

$$\frac{16}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{x^3}{3}\Big|_{-1}^{1} - \frac{1}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

In conclusion:

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots = \frac{\pi^4}{96}$$