Calculus - Lecture 6

Integrals and primitives.

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The Riemann-Darboux integral

A partition P of the interval $[a,b]$ is a finite set of points $\{x_0, x_1, .., x_n\}$ satisfying

$$a = x_0 < x_1 < .. < x_n = b.$$

Consider a bounded function f defined on $[a,b]$.

Riemann sum of f related to P:

$$R_f(P) = \sum_{i=1}^{n} f(x_i^\star)(x_i - x_{i-1}) \quad \text{where} \quad x_i^\star \in [x_{i-1}, x_i]$$

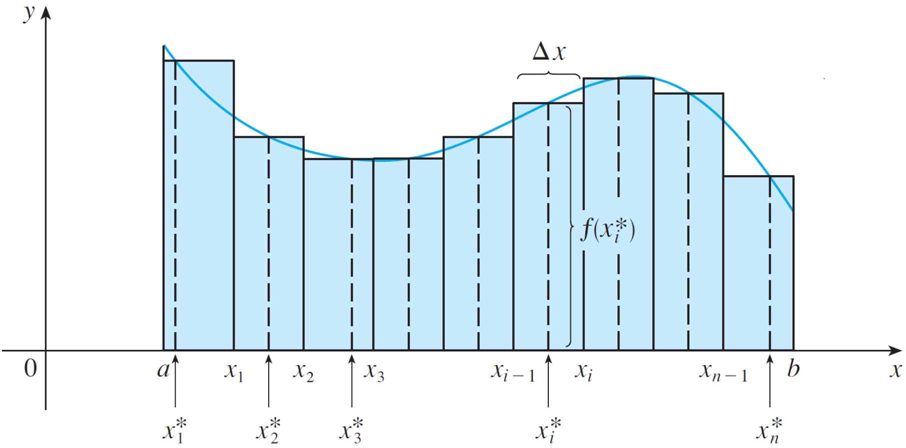
Upper Darboux sum of f related to P:

$$U_f(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{where} \quad M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$$

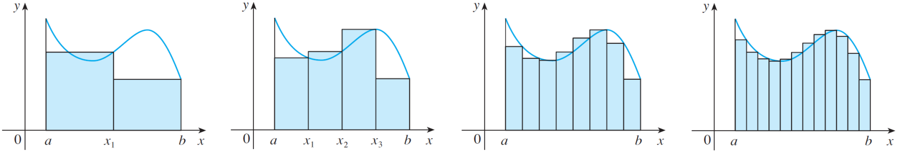
Lower Darboux sum of f related to P:

$$L_f(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \text{where} \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$$

The Riemann-Darboux integral



The Riemann-Darboux integral



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

The Riemann-Darboux integral

Consider

$$M = \sup\{f(x) \mid a \le x \le b\} \quad \text{and} \quad m = \inf\{f(x) \mid a \le x \le b\}.$$

For any partition P of $[a, b]$ we have:

$$m(b-a) \le L_f(P) \le R_f(P) \le U_f(P) \le M(b-a).$$

The following sets are bounded:

$$L_f = \{L_f(P) \mid P \text{ is a partition of } [a, b]\}$$

$$U_f = \{U_f(P) \mid P \text{ is a partition of } [a, b]\}$$

The Riemann-Darboux integral

So $\mathcal{L}_f = \sup L_f$ and $\mathcal{U}_f = \inf U_f$ exist. Moreover:

$$\mathcal{L}_f \leq \mathcal{U}_f.$$

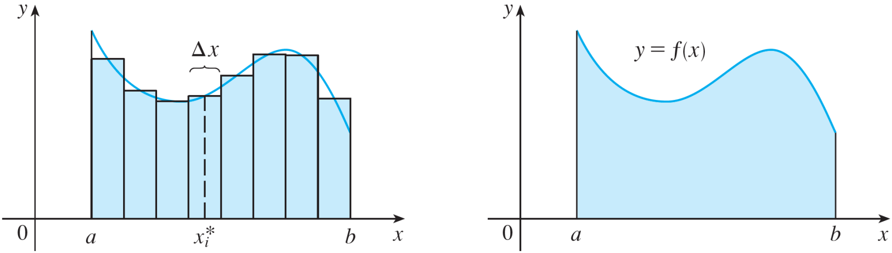
A bounded function defined on $[a, b]$ is Riemann-Darboux integrable on $[a, b]$ if

$$\mathcal{L}_f = \mathcal{U}_f.$$

This common value is denoted by

$$\int_a^b f(x)\,dx = \mathcal{L}_f = \mathcal{U}_f = \lim_{n\to\infty} R_f(P).$$

The Riemann-Darboux integral



Example

Evaluate the Riemann sum for $f(x) = x^3 - x$ over the interval $[0, 3]$, taking the sample points $x_i^\star$ to be the right endpoints of the intervals $[x_{i-1}, x_i]$, where $x_i = \frac{3i}{n}$. Compute $\int_0^3 f(x)dx$.

$$\begin{aligned} R_f(P) &= \sum_{i=1}^{n} f(x_i^\star)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - \frac{3i}{n}\right] \\ &= \frac{3^4}{n^4}\sum_{i=1}^{n} i^3 - \frac{3^2}{n^2}\sum_{i=1}^{n} i = \frac{3^4}{n^4}\frac{n^2(n+1)^2}{4} - \frac{3^2}{n^2}\frac{n(n+1)}{2} \\ &= \frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{9}{2}\left(1+\frac{1}{n}\right) \overset{n\to\infty}{\longrightarrow} \frac{81}{4} - \frac{9}{2} = \frac{63}{4}. \end{aligned}$$

Therefore:

$$\int_0^3 f(x)dx = \frac{63}{4}.$$

Properties of the Riemann-Darboux integral

If f and g are Riemann-Darboux integrable on $[a,b]$ then all the integrals below exist and

$$\int_a^b (\alpha\, f(x) + \beta\, g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx \text{ for any } \alpha, \beta \in \mathbb{R}.$$

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \text{ for any } a \leq c \leq b.$$

$$\text{If } f(x) \leq g(x) \text{ on } [a,b] \text{ then } \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Classes of Riemann-Darboux integrable functions

If f is continuous on $[a, b]$, then f is Riemann-Darboux integrable on $[a, b]$.

A function f is called piecewise continuous on $[a, b]$ if there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ and continuous functions f_i defined on $[x_{i-1}, x_i]$, such that $f(x) = f_i(x)$ for $x \in (x_{i-1}, x_i)$, $i = 1, 2, \dots, n$.

A piecewise continuous function is Riemann-Darboux integrable and

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_i(x)\,dx.$$

The integral mean value theorem

If f and g are continuous on $[a, b]$ and $g(x) \geq 0$ for $x \in [a, b]$, then there exists an intermediate point c between a and b such that

$$\int_a^b f(x) \cdot g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

The Fundamental Theorem of Calculus

Theorem

If $f : [a, b] \to \mathbb{R}$ *is Riemann-Darboux integrable on* $[a, b]$ *and*

$$F(x) = \int_a^x f(t)\,dt$$

then F *is continuous on* $[a, b]$*.*

Furthermore, if f *is continuous on* $[a, b]$*, then* F *is differentiable on* $[a, b]$ *and*

$$F' = f.$$

Any function Φ such that $\Phi' = f$ is called a primitive (antiderivative) of f.

- Two primitives of the same function f differ by a constant.
- If F is a primitive of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Integration by parts

If the functions f and g are continuously differentiable on $[a,b]$, then

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

where $\int f(x)g'(x)dx$ and $\int f'(x)g(x)dx$ represent the set of primitives of the functions fg' and $f'g$, respectively.

Consequence for definite integrals:

$$\int_a^b f(x)\cdot g'(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)\cdot g(x)\,dx.$$

Change of variables (substitution)

If the function $g : [\alpha, \beta] \to [a, b]$ is a continuously differentiable bijection with the property $g(\alpha) = a$, $g(\beta) = b$ and $f : [a, b] \to \mathbb{R}^1$ is continuous, then

$$\left(\int f(x)\, dx \right) \circ g = \int (f \circ g)(t) \cdot g'(t)\, dt$$

where $\int f(x)dx$ and $\int (f \circ g)(t) \cdot g'(t)\, dt$ represent the set of primitives of the functions f and $(f \circ g) \cdot g'$, respectively.

Consequence for definite integrals:

$$\int_{g(\alpha)}^{g(\beta)} f(x)\, dx = \int_{\alpha}^{\beta} (f \circ g)(t) \cdot g'(t)\, dt$$

Improper integrals of type I (infinite interval)

Definition

Consider a bounded function $f : [a, \infty) \to \mathbb{R}$ which is Riemann-Darboux integrable on every closed interval of the form $[a, b]$, with $b > a$.

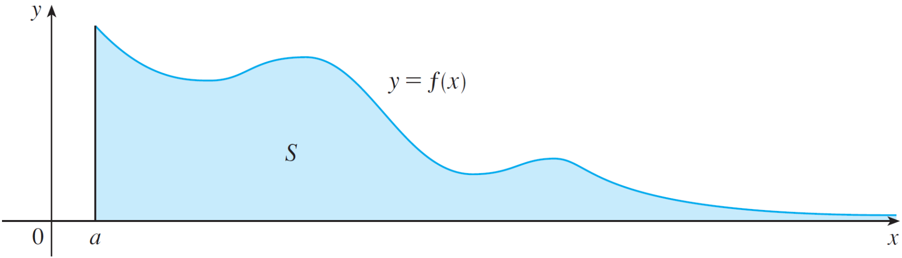
The improper integral of type I $\int_a^\infty f(x)dx$ is called convergent if the limit

$$\lim_{b\to\infty} \int_a^b f(x)\, dx$$

exists (as a finite number). In this case: $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$

If the above limit does not exist (or is infinite), the improper integral is called divergent.

Improper integrals of type I (infinite interval)



Improper integrals of type I (infinite interval)

In a similar manner, the improper integral of type I over the interval $(-\infty, b]$ is defined as:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx$$

(provided that the above limit exists, as a finite number).

If both improper integrals $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx.$$

Improper integrals of type I (infinite interval)

Example. Study the convergence of the integral $\int_1^\infty \frac{1}{x^p}dx$, where $p > 0$.

Assume that $p \neq 1$ and let $b > 1$. We have:

$$\int_1^b \frac{1}{x^p}dx = \left.\frac{x^{1-p}}{1-p}\right|_1^b = \frac{b^{1-p}-1}{1-p} .$$

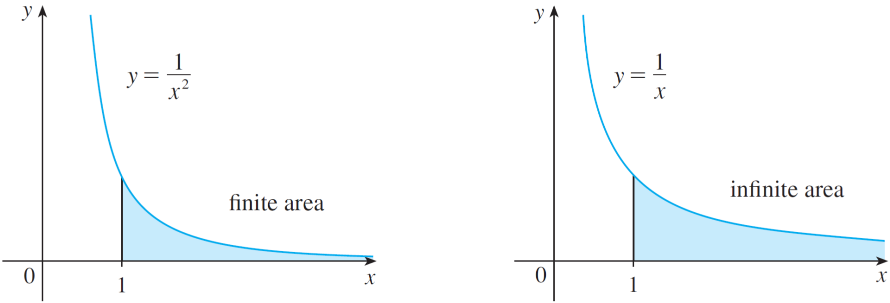
On the other hand, if $p = 1$, we have:

$$\int_1^b \frac{1}{x}dx = \ln(x)|_1^b = \ln(b) .$$

We take the limit as $b \to \infty$ in the above relations:

- if $p \in (0, 1]$ we have $\lim_{b\to\infty} \int_1^b \frac{1}{x^p}dx = \infty \implies \int_1^\infty \frac{1}{x^p}dx$ is divergent
- if $p \in (1, \infty)$ we have $\lim_{b\to\infty} \int_1^b \frac{1}{x^p}dx = \frac{1}{p-1} \implies \int_1^\infty \frac{1}{x^p}dx$ is convergent

Improper integrals of type I (infinite interval)



Improper integrals of type II (unbounded functions)

Definition

Consider the function $f : (a, b] \to \infty$ with a vertical asymptote at a, which is Riemann-Darboux integrable on any interval $[a+\epsilon, b]$, with $\epsilon \in (0, b-a)$.

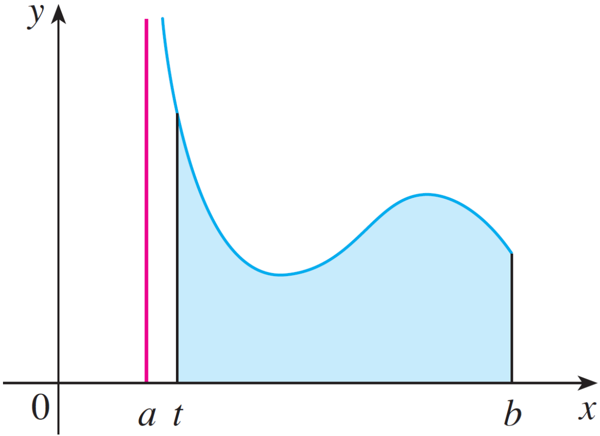
The improper integral of type II $\int_a^b f(x)\,dx$ is called convergent if the limit

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x)\,dx$$

exists (as a finite number). In this case: $\int_a^b f(x)dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x)\,dx.$

If the above limit does not exist (or is infinite), the improper integral is called divergent.

Improper integrals of type II (unbounded functions)



Improper integrals of type II (unbounded functions)

In a similar manner, the improper integral of type II over the interval $[a, b)$ is defined as:

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\, dx$$

(provided that the above limit exists, as a finite number).

If the function f has a vertical asymptote at $c \in (a, b)$ and if both improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Improper integrals of type II (unbounded functions)

Example. Study the convergence of the integral $\int_0^1 \frac{1}{x^p} dx$, where $p > 0$.

Assume that $p \neq 1$ and let $\epsilon > 0$. We have:

$$\int_\epsilon^1 \frac{1}{x^p} dx = \left. \frac{x^{1-p}}{1-p} \right|_\epsilon^1 = \frac{1-\epsilon^{1-p}}{1-p} .$$

On the other hand, if $p = 1$, we have:

$$\int_\epsilon^1 \frac{1}{x} dx = \ln(x)|_\epsilon^1 = -\ln(\epsilon) .$$

We take the limit as $\epsilon \to 0^+$ in the above relations:

- if $p \in [1, \infty)$ we have $\lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^p} dx = \infty \implies \int_0^1 \frac{1}{x^p} dx$ is divergent
- if $p \in (0, 1)$ we have $\lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^p} dx = \frac{1}{1-p} \implies \int_0^1 \frac{1}{x^p} dx$ is convergent

Example summary

Let $a > 0$.

The improper integral of type I:

$$\int_a^\infty \frac{1}{x^p}dx \text{ is convergent } \iff p \in (1, \infty)$$

The improper integral of type II:

$$\int_0^a \frac{1}{x^p}dx \text{ is convergent } \iff p \in (0, 1)$$

Comparison test for improper integrals

Theorem

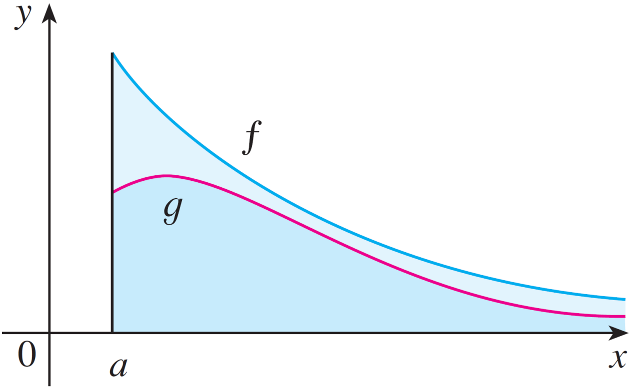
Suppose that the functions f and g defined on $[a, \infty)$ are Riemann-Darboux integrable on every closed interval of the form $[a, b]$, with $b > a$, and

$$0 \leq g(x) \leq f(x) \quad , \forall x \geq a$$

- *If $\int_a^\infty f(x)\,dx$ is convergent then $\int_a^\infty g(x)\,dx$ is convergent.*
- *If $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.*

Remark: A similar theorem is true for improper integrals of type II as well.

Comparison test for improper integrals



Comparison test for improper integrals

Example. Show that the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is convergent.

The following inequality holds (check using derivatives!):

$$e^x \geq x+1 \quad , \forall\, x \geq 0.$$

Therefore:

$$0 \leq e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{x^2+1} \quad , \forall\, x \in \mathbb{R}.$$

The following type I improper integral is convergent:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \arctan(x)\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Based on the comparison test, it follows that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is also convergent.

(in fact, it's exact value is $\sqrt{\pi}$, shown by multivariable calculus methods)