Calculus - Lecture 5

Power Series.
Taylor Polynomials and Taylor Series.

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Power series

A series of functions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

is called power series.

The general term of this series is the function

$$f_n(x) = a_n x^n$$

where (a_n) is sequence of real numbers, called sequence of coefficients.

Variation: power series centered at the point x_0:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

The Abel-Cauchy-Hadamard theorem

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$.

Denoting

$$\omega = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}} \frac{|a_{n+1}|}{|a_n|} \in [0, +\infty]$$

and the radius of convergence

$$R = \begin{cases} \frac{1}{\omega}, & \text{if } \omega \neq 0 \\ +\infty, & \text{if } \omega = 0 \end{cases}$$

the power series is

- absolutely convergent for $|x| < R$.
- divergent for $|x| > R$.
- uniformly convergent on any closed interval $[-r, r] \subset (-R, R)$.

!!! this theorem does not provide information about convergence at $x = \pm R$.

Example

Let us consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

The sequence of coefficients is $a_n = \frac{1}{n(n+1)}$. We compute:

$$\omega = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n(n+1)}{(n+1)(n+2)} = 1 \quad \Longrightarrow \quad R = \frac{1}{\omega} = 1$$

- The power series is absolutely convergent for $|x| < 1$.
- The power series is divergent for $|x| > 1$.

For $x = 1$, we have $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \to$ convergent! (harmonic series).

For $x = -1$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \to$ convergent! (alternating series).

The set of convergence is $[-1, 1]$.

Arithmetics of power series

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ with radii of convergence R_1 and R_2, where $0 \leq R_1 \leq R_2$.

Then the following power series have radii of convergence at least R_1:

- the sum $\sum_{n=0}^{\infty} (a_n + b_n) x^n$
- the scalar product $\sum_{n=0}^{\infty} k \cdot a_n x^n$
- the Cauchy product $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Arithmetics of power series

Moreover, if we denote the sums of the two series by

$$\sum_{n=0}^{\infty} a_n x^n = f(x) \quad \text{and} \quad \sum_{n=0}^{\infty} b_n x^n = g(x),$$

then, for any $x \in (-R_1, R_1)$, we have:

- $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n) x^n = f(x) + g(x)$
- $\displaystyle\sum_{n=0}^{\infty} k \cdot a_n x^n = k \cdot f(x)$
- $\displaystyle\sum_{n=0}^{\infty} c_n x^n = f(x) g(x)$

Continuity and differentiability of a power series' sum

Theorem (Continuity)

The sum $f(x)$ of the power series $\sum_{n=0}^{\infty} a_n x^n$ is a continuous function on $(-R, R)$, where R denotes the radius of convergence.

Theorem (Differentiability)

The sum $f(x)$ of the power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$ is k-times differentiable, for any $k \in \mathbb{N}$ and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\dots(n-k+1)x^{n-k} \quad , \forall\, x \in (-R, R).$$

For $x = 0$, it follows that $f^{(k)}(0) = a_k \cdot k! \implies a_k = \frac{f^{(k)}(0)}{k!}$

Continuity and differentiability of a power series' sum

Therefore, the following representation is valid:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad , \forall\, x \in (-R, R).$$

Hence, if $|x|$ is sufficiently small, we can estimate

$$f(x) \simeq \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n.$$

Similarly, for power series centered at x_0, we have:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad , \forall\, x \in (x_0 - R, x_0 + R).$$

Hence, if x is sufficiently close to x_0, we can estimate

$$f(x) \simeq \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Example

Considering $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$, for $x \in [-1, 1]$, we have:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x^n)'}{n(n+1)} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1}.$$

Hence, multiplying by x^2 and differentiating once more, we obtain:

$$\left(x^2 f'(x)\right)' = \left(\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1}\right)' = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} = \frac{1}{1-x} - 1 \quad , \forall\, x \in (-1, 1).$$

Integrating, we obtain:

$$x^2 f'(x) = \int \left(\frac{1}{1-x} - 1\right) dx = -\ln(1-x) - x$$

and therefore:

$$f(x) = -\int \left(\frac{\ln(1-x)}{x^2} + \frac{1}{x}\right) = \frac{(1-x)\ln(1-x)}{x} + 1 \quad , \forall\, x \in (-1, 1).$$

Taylor polynomials

Motivating Problem. Find a practical method for calculating e^x, $\sin(x)$, etc. Often you have an accuracy in mind (e.g. "to five decimal places").

Let f be an n-times continuously differentiable function on an open interval containing the point a.

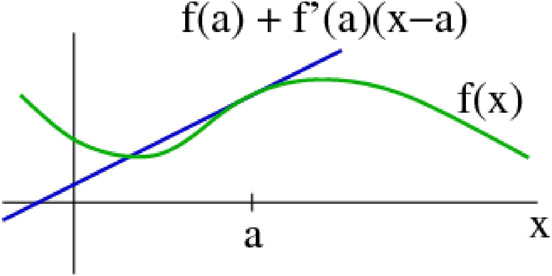
The n-th degree Taylor polynomial to the function f at the point a is defined by:

$$T_{n,a}f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Question. Can we approximate the value of the function f at a point x by the value of the Taylor polynomial $T_{n,a}f(x)$?

Taylor polynomials - simplest cases

- $T_{0,a}f(x) = f(a) \to$ a constant;
- linear approximation: $T_{1,a}f(x) = f(a) + f'(a)(x-a)$



Taylor polynomials - example

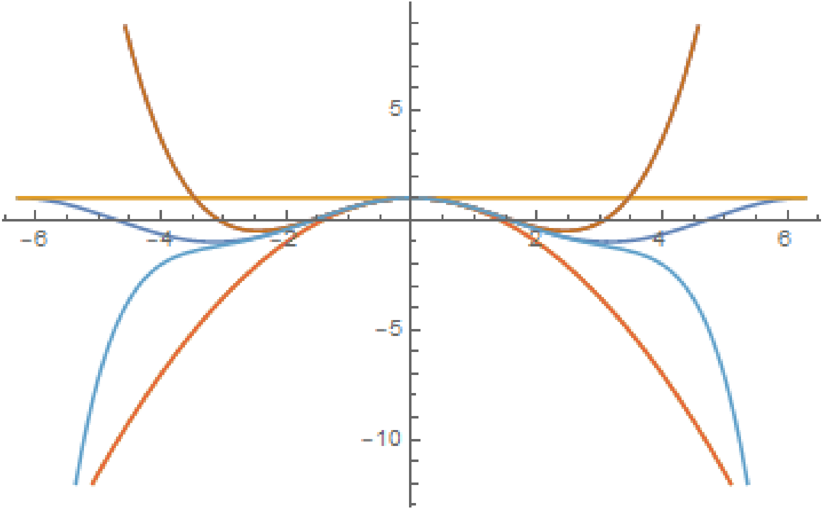
Find the Taylor polynomial of order $n = 6$ of the function $f(x) = \cos(x)$ at the point $a = 0$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
5	$-\sin(x)$	0
6	$-\cos(x)$	-1

$$T_{6,0}f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(6)}(0)}{6!}x^6 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

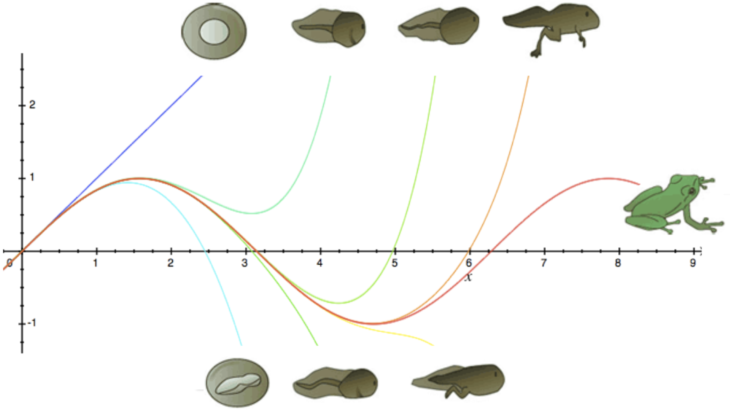
Taylor polynomials - example

Taylor polynomials for $f(x) = \cos(x)$, up to order $n = 6$



Applications

Numerical approximation of functions.
(If you want a tadpole, do you need the DNA for the entire frog?)



Applications

Example: Approximate $\cos(0.1)$.

An approximate value is given by the Taylor polynomial of order 6:

$$T_{6,0}(0.1) = 1 - \frac{(0.1)^2}{2!} + \frac{(0.1)^4}{4!} - \frac{(0.1)^6}{6!} = 0.995004$$

What is the accuracy of this approximation?

The first remainder theorem

Theorem

Let f be $(n+1)$ times continuously differentiable on an open interval containing the points a and x. Then the difference between $f(x)$ and $T_{n,a}f(x)$ is given by

$$f(x) - T_{n,a}f(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

for some c between a and x.

The error in approximating $f(x)$ by the value of the polynomial $T_{n,a}f(x)$ is the remainder term:

$$R_{n,a}f(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Taylor's formula of degree n:

$$f(x) = T_{n,a}f(x) + R_{n,a}f(x)$$

The first remainder theorem: application

Example: Find the error of approximating $\cos(0.1)$ by $T_{6,0}(0.1) = 0.995004$.

$$R_{6,0}f(x) = \frac{x^7}{7!} f^{(7)}(c) = \frac{x^7}{7!} \sin(c)$$

Hence:

$$R_{6,0}f(0.1) = \frac{(0.1)^7}{7!} \sin(c) \quad \text{where } c \in (0, 0.1)$$

Therefore, the absolute error of approximation is:

$$\epsilon_a = |R_{6,0}f(0.1)| \leq \frac{(0.1)^7}{7!} = \frac{10^{-7}}{5040} \leq \frac{10^{-7}}{5000} = 2 \cdot 10^{-11}$$

$\rightarrow$ accuracy of at least 10 digits

Taylor series representation

Suppose that the function f has derivatives of all orders on some interval containing the point a and also that

$$\lim_{n\to\infty} R_{n,a}(x) = 0$$

for each x in that interval. Then for any x in that interval, we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called Taylor series representation of the function $f(x)$ at the point a.

When $a = 0$, the above series is also called MacLaurin series.

Taylor series representation: examples

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^k \frac{x^{2k}}{(2k)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \ , \ \forall x \in \mathbb{R}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \ , \ \forall x \in \mathbb{R}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ , \ \forall x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \ , \ x \in (-1, 1]$$