Calculus - Lecture 12

Line integrals. Green's theorem.

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Elementary curves in $\mathbb{R}^2$

An elementary curve in $\mathbb{R}^2$ is a set of points $C \subset \mathbb{R}^2$ for which there exists a smooth (class C^1) function $\varphi : [a, b] \to C$ which is bijective on $[a, b)$.

The points $A = \varphi(a)$ and $B = \varphi(b)$ are called the end points of the curve.

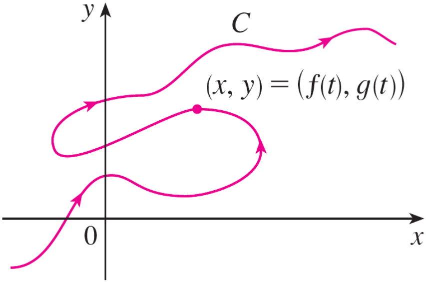
The function φ is called a parametric representation of the curve.

The vector $\varphi'(t)$ is tangent to the curve at the point $\varphi(t)$.

Parametric equations:

$$C : \begin{cases} x = f(t) \\ y = g(t) \end{cases} , \; t \in [a, b]$$

where f and g are the scalar components of φ.



Elementary curves in $\mathbb{R}^2$

An elementary closed curve is a curve with parametric representation φ s.t.

$$\varphi(a) = \varphi(b).$$

Remarks:

- Any elementary curve has an infinity of parametric representations.
- The end points of an elementary curve are independent of the parametric representation of the curve.

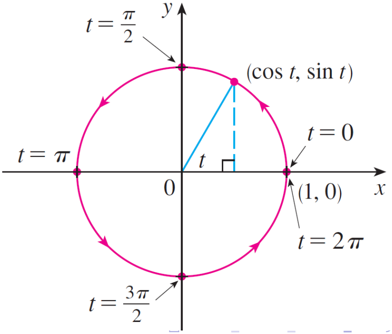
Example: A parametric representation of the circle $x^2 + y^2 = 1$ is

$$\varphi : [0, 2\pi] \to \mathbb{R}^2, \quad \varphi(t) = (\cos t, \sin t).$$

Parametric equations:

$$C : \begin{cases} x = \cos t \\ y = \sin t \end{cases}, \; t \in [0, 2\pi]$$

Closed curve: $\varphi(0) = \varphi(2\pi) = (1, 0)$.



The length of a curve in $\mathbb{R}^2$

The length of the elementary curve $C \subset \mathbb{R}^2$ with parametric representation $\varphi : [a,b] \to \mathbb{R}^2$ is given by:

$$l = \int_a^b \|\varphi'(t)\|\,dt = \int_a^b \sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}\,dt$$

where $\varphi = (\varphi_1, \varphi_2)$.

Remark: The curve length is independent of the parametric representation of the curve C!

The arc length of the elementary curve C with representation φ is defined as

$$ds = \|\varphi'(t)\|\,dt = \sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}\,dt$$

In the previous **Example:** the length of the circle $x^2 + y^2 = 1$ is

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \int_0^{2\pi} 1 dt = 2\pi.$$

Line integral with respect to arc length in $\mathbb{R}^2$

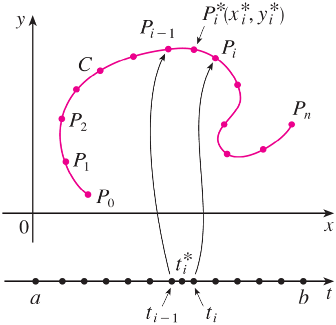
Consider a curve $C \subset \mathbb{R}^2$ with the parametric representation $\varphi : [a,b] \to C$ and a continuous function $f : C \subset \mathbb{R}^2 \to \mathbb{R}$.

Line integral of first type (with respect to arc length):

$$\int_C f(x,y)ds = \int_a^b f(\varphi_1(t), \varphi_2(t))\sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}\, dt$$

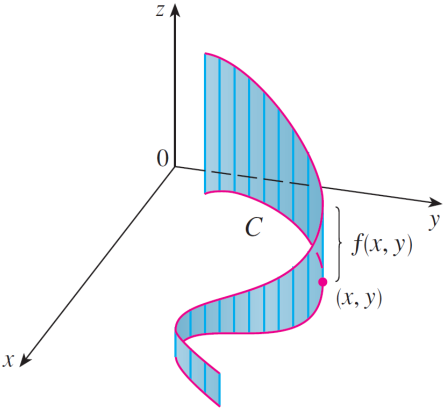
- divide the interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$, of equal length
- consider $P_i = \varphi(t_i) \in C$
- the points P_i divide the curve C into n subarcs with lengths Δs_1, Δs_2, ..., Δs_n
- choose an intermediate point $P_i^*(x_i^*, y_i^*) = \varphi(t_i^*)$, with $t_i^* \in [t_{i-1}, t_i]$

$$\Longrightarrow \int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$



Interpretation

The line integral of a positive function $f(x,y) \geq 0$, $(x,y) \in C$ represents the area of one side of the "curtain" in the figure below, whose base is C and whose height above the point (x,y) is $f(x,y)$.

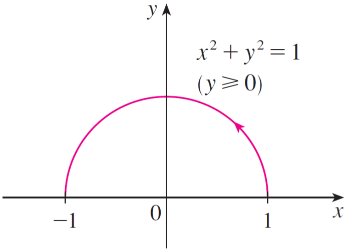


Example

Compute $\int_C (2+x^2y)ds$ where C si the upper half of the unit circle $x^2+y^2=1$.

Considering the following parametric equations of the curve C:

$$C : \begin{cases} x = \cos t \\ y = \sin t \end{cases} , \ t \in [0,\pi]$$



as the arc length is

$$ds = \sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}dt = dt$$

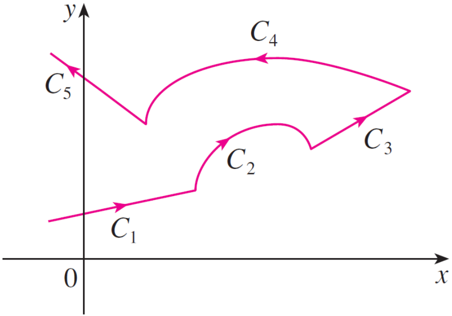
we obtain:

$$\int_C (2+x^2y)ds = \int_0^\pi (2+(\cos t)^2 \sin t)dt = 2\pi - \frac{1}{3}(\cos t)^3\Big|_0^\pi = 2\pi + \frac{2}{3}.$$

Line integrals on piecewise smooth curves

Assume that the curve C is the union of smooth curves $C_1, C_2, ..., C_n$ where the initial point of C_i is the terminal point of C_{i-1}, for $i = \overline{1, n}$.

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + ... + \int_{C_n} f(x,y)ds$$



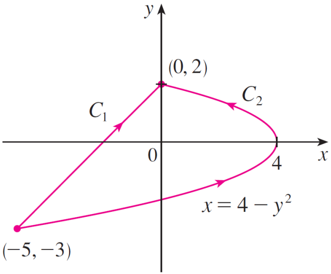
Line integrals with respect to coordinate variables

Line integrals of f with respect to x and y:

$$\int_C f(x,y)dx = \int_a^b f(\varphi_1(t),\varphi_2(t))\varphi_1'(t)\,dt$$

$$\int_C f(x,y)dy = \int_a^b f(\varphi_1(t),\varphi_2(t))\varphi_2'(t)\,dt$$

Example. Let C_2 be the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.



$$C_2 : \begin{cases} x = 4 - t^2 \\ y = t \end{cases}, \; t \in [-3, 2]$$

$$\Longrightarrow \int_{C_2} (y^2 dx + x dy) = \int_{-3}^{2} t^2(-2t)dt + \int_{-3}^{2} (4-t^2)dt = \int_{-3}^{2} (-2t^3+4-t^2)dt = \ldots$$

Line integrals in $\mathbb{R}^3$

Consider a smooth curve $C \subset \mathbb{R}^3$ with the parametric representation $\varphi : [a,b] \to \mathbb{R}^3$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, and a continuous function $f : C \to \mathbb{R}$.

$$\int_C f(x,y,z)ds = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2 + \varphi_3'(t)^2}\, dt$$

$$\int_C f(x,y,z)dx = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_1'(t)\, dt$$

$$\int_C f(x,y,z)dy = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_2'(t)\, dt$$

$$\int_C f(x,y,z)dz = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_3'(t)\, dt$$

Example

C is the circular helix with the parametric equations

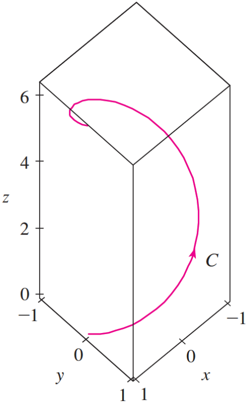
$$C : \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}, \quad t \in [0, 2\pi]$$

$$\int_C y \sin z ds = \int_0^{2\pi} (\sin t)^2 \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} dt$$

$$= \sqrt{2} \int_0^{2\pi} \sin^2 t \; dt$$

$$= \frac{\sqrt{2}}{2} \int_0^{2\pi} (1 - \cos(2t)) dt$$

$$= \frac{\sqrt{2}}{2} \left(2\pi - \frac{\sin(2t)}{2} \Big|_0^{2\pi} \right) = \sqrt{2}\pi.$$



Green's theorem

Theorem (Green's theorem)

Let D be a closed bounded region in the plane whose boundary is a piecewise smooth closed curve C.

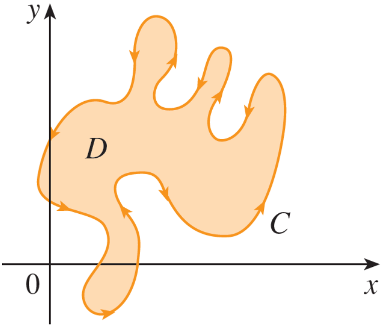
If f and g are functions of class C^1 on an open region containing D, then

$$\iint\limits_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy = \oint_C f\, dx + g\, dy$$

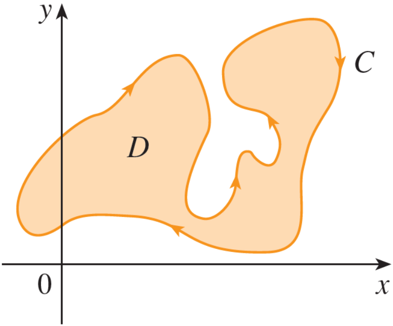
where the positive orientation is considered for the curve C.

Positive vs. negative orientation

positive orientation (counterclockwise)



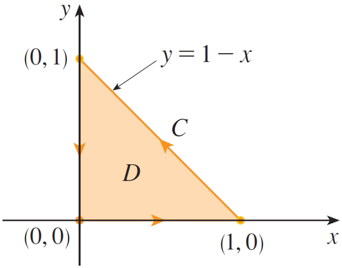
negative orientation (clockwise)



Example

C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

Compute the integral: $\oint_C x^4 dx + xy dy$.



$$f(x,y) = x^4 \quad \Longrightarrow \quad \frac{\partial f}{\partial y} = 0$$

$$g(x,y) = xy \quad \Longrightarrow \quad \frac{\partial g}{\partial x} = y$$

$$D = \{(x,y) \in \mathbb{R}^2 : x \in [0,1],\ y \in [0, 1-x]\}$$

$$\oint_C x^4 dx + xy dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \iint_D y \, dx dy =$$

$$= \int_0^1 \int_0^{1-x} y \, dy \, dx = \int_0^1 \frac{1}{2} y^2 \Big|_{y=0}^{y=1-x} dx = \frac{1}{2} \int_0^1 (1-x)^2 dx =$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1 = \frac{1}{6}$$