Calculus - Lecture 11

Double integrals.

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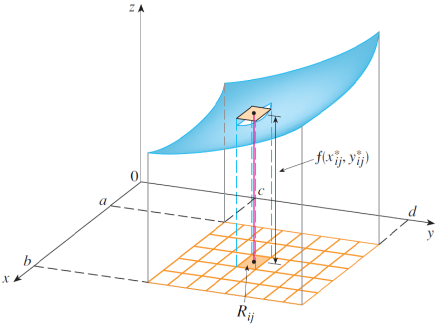
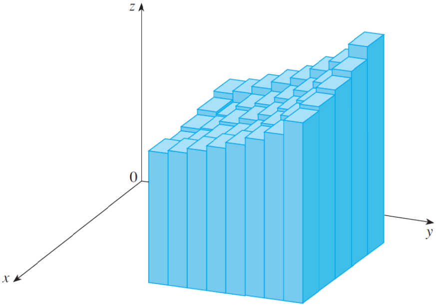
Double integrals





Jordan measurable sets in $\mathbb{R}^2$

Consider the set of bounded intervals I of the form

$$(a,b),\ [a,b),\ (a,b],\ [a,b], \quad \text{where } a,b \in \mathbb{R}.$$

The cartesian product $\Delta = I_1 \times I_2$ is a rectangle in $\mathbb{R}^2$.

The area of such a rectangle Δ is defined by

$$\text{area}(\Delta) = \text{length}(I_1) \cdot \text{length}(I_2).$$

Consider the set $\mathcal{P}$ of all finite reunions of rectangles Δ:

$$P \in \mathcal{P} \quad \text{iff.} \quad \exists\ \Delta_1, \Delta_2, ..., \Delta_n \text{ s. t. } P = \bigcup_{i=1}^{n} \Delta_i.$$

- If $P_1, P_2 \in \mathcal{P}$, then $P_1 \cup P_2 \in \mathcal{P}$ and $P_1 \setminus P_2 \in \mathcal{P}$.
- Any $P \in \mathcal{P}$ can be written as the union of disjoint rectangles $\Delta_1, \Delta_2, ..., \Delta_n$ ($\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$):

$$P = \bigcup_{i=1}^{n} \Delta_i$$

Jordan measurable sets in $\mathbb{R}^2$

The area of a set $P \in \mathcal{P}$ is

$$\text{area}(P) = \sum_{i=1}^{n} \text{area}(\Delta_i), \quad \text{where } P = \bigcup_{i=1}^{n} \Delta_i \text{ and } \Delta_1, \Delta_2, ..., \Delta_n \text{ are disjoint.}$$

The area defined in this way satisfies:

- $\text{area}(P) > 0$ for any $P \in \mathcal{P}$.
- if $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 = \emptyset$, then

$$\text{area}(P_1 \cup P_2) = \text{area}(P_1) + \text{area}(P_2).$$

- $\text{area}(P)$ is independent on the decomposition of the set P in a finite union of disjoint rectangles.

Jordan measurable sets in $\mathbb{R}^2$

For a bounded set $A \subset \mathbb{R}^2$, we define

$$\text{area}_i(A) = \sup_{P \subset A, P \in \mathcal{P}} \text{area}(P) \quad \text{and} \quad \text{area}_e(A) = \inf_{P \supset A, P \in \mathcal{P}} \text{area}(P)$$

A bounded set $A \subset \mathbb{R}^2$ is called Jordan measurable if

$$\text{area}_i(A) = \text{area}_e(A).$$

The area of a Jordan measurable set $A \subset \mathbb{R}^2$ is defined as

$$\text{area}(A) = \text{area}_i(A) = \text{area}_e(A)$$

- If A_1, A_2 are Jordan measurable, then so are $A_1 \cup A_2$ and $A_1 \setminus A_2$.
- If $A_1 \cap A_2 = \emptyset$, then

$$\text{area}(A_1 \cup A_2) = \text{area}(A_1) + \text{area}(A_2).$$

Riemann-Darboux integral of two variable functions

Consider a bounded and Jordan measurable set $A \subset \mathbb{R}^2$.

A partition P of A is a finite set of disjoint Jordan measurable subsets A_i, $i = \overline{1,n}$ of A satisfying:

$$\bigcup_{i=1}^{n} A_i = A.$$

The diameter of a set A_i is

$$d(A_i) = \max_{(x',y'),(x'',y'')\in A_i} \sqrt{(x' - x'')^2 + (y' - y'')^2}$$

The norm of the partition P is

$$\nu(P) = \max\{d(A_1), d(A_2), \cdots, d(A_n)\}.$$

Darboux and Riemann sums

Let $f : A \to \mathbb{R}^1$ be a bounded function. Then f is bounded on each part A_i and has a least upper bound M_i and a greatest lower bound m_i on A_i.

The upper Darboux sum of f with respect to the partition P is

$$U_f(P) = \sum_{i=1}^{n} M_i \cdot \text{area}(A_i), \quad \text{where } M_i = \sup\{f(x,y) \,|\, (x,y) \in A_i\}.$$

The lower Darboux sum of f with respect to the partition P is

$$L_f(P) = \sum_{i=1}^{n} m_i \cdot \text{area}(A_i), \quad \text{where } m_i = \inf\{f(x,y) \,|\, (x,y) \in A_i\}.$$

The Riemann sum of f with respect to the partition P is

$$\sigma_f(P) = \sum_{i=1}^{n} f(\xi_i, \eta_i) \cdot \text{area}(A_i) \quad \text{where } (\xi_i, \eta_i) \in A_i.$$

The following inequalities hold

$$L_f(P) \leq \sigma_f(P) \leq U_f(P).$$

Riemann-Darboux integral of two variable functions

Consider the numbers m and M such that $m \leq f(x,y) \leq M$ for all $(x,y) \in A$.

$$m \cdot \text{area}(A) = m \cdot \sum_{i=1}^{n} \text{area}(A_i) \leq L_f(P) \leq U_f(P) \leq M \cdot \sum_{i=1}^{n} \text{area}(A_i) = M \cdot \text{area}(A)$$

Hence, the following sets are bounded:

$$L_f = \{L_f(P) \mid P \text{ is a partition of } A\}$$
$$U_f = \{U_f(P) \mid P \text{ is a partition of } A\}$$

We can therefore consider $\mathcal{L}_f = \sup_P L_f$ and $\mathcal{U}_f = \inf_P U_f$.

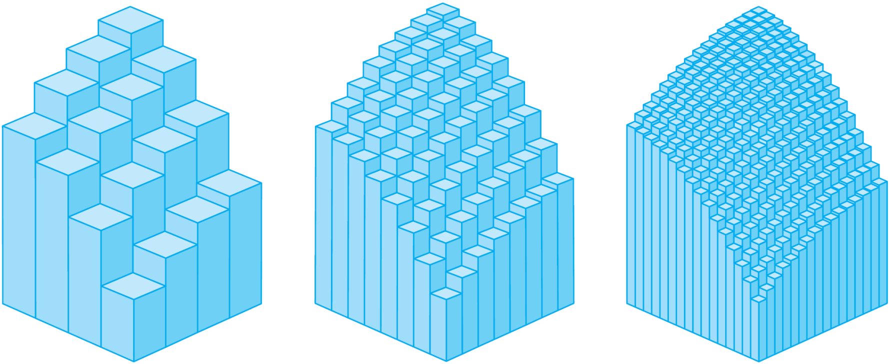
If the function f is defined and bounded on A, then

$$\mathcal{L}_f \leq \mathcal{U}_f.$$

The function f is Riemann-Darboux integrable on A if

$$\mathcal{L}_f = \mathcal{U}_f := \underbrace{\iint_A f(x,y)\,dx\,dy}_{\text{double integral of } f \text{ on } A}$$

Riemann-Darboux integral of a two-variable function



Classes of Riemann-Darboux integrable functions

If f is continuous a Jordan measurable set A, then f is Riemann-Darboux integrable on A.

A function f is called piecewise-continuous on A if there exists a partition $P = \{A_1, \cdots, A_n\}$ of A and continuous functions f_i, $i = \overline{1, n}$ defined on A_i such that $f(x) = f_i(x)$ for $x \in Int(A_i)$.

A piecewise-continuous function is Riemann-Darboux integrable and

$$\iint\limits_{A} f(x,y)\,dx\,dy = \sum_{i=1}^{n} \iint\limits_{A_i} f_i(x,y)\,dx\,dy.$$

Properties of the Riemann-Darboux integral

If f and g are Riemann-Darboux integrable on A, then all the integrals below exist and the following hold:

$$\iint\limits_A [\alpha f(x,y) + \beta g(x,y)]\, dxdy = \alpha \iint\limits_A f(x,y)dxdy + \beta \iint\limits_A g(x,y)dxdy,\ \forall \alpha, \beta \in \mathbb{R}^1$$

$$\iint\limits_A f(x,y)dxdy = \iint\limits_{A_1} f(x,y)dxdy + \iint\limits_{A_2} f(x,y)dxdy \text{ where } A_1 \cup A_2 = A,\ A_1 \cap A_2 = \emptyset$$

$$\text{if } f(x,y) \leq g(x,y) \text{ on } A, \text{ then } \iint\limits_A f(x,y)\, dx\, dy \leq \iint\limits_A g(x,y)\, dx\, dy$$

The mean value theorem:

If $f : A \to \mathbb{R}^1$ is integrable on A and $m \leq f(x,y) \leq M$ for any $(x,y) \in A$, then:

$$m \cdot \text{area}(A) \leq \iint\limits_A f(x,y)\, dx\, dy \leq M \cdot \text{area}(A).$$

Double integral on a rectangle

Theorem (Fubini's Theorem)

Assume that A is a rectangle, $A = [a, b] \times [c, d]$ and $f : A \to \mathbb{R}^1$ is a continuous function. Then:

$$\iint\limits_A f(x, y)\, dx\, dy = \int\limits_a^b \left(\int\limits_c^d f(x, y)\, dy \right) dx = \int\limits_c^d \left(\int\limits_a^b f(x, y)\, dx \right) dy$$

$\Longrightarrow$ the computation of a double integral on a rectangular domain reduces to the computation of two successive (or *iterated*) single-variable integrals.

Example. If $A = [0, 2] \times [1, 3]$ then

$$\iint_A (x - 3y^2) dx dy = \int_0^2 \int_1^3 (x - 3y^2) dy\ dx = \int_0^2 (xy - y^3)\Big|_{y=1}^{y=3} dx$$

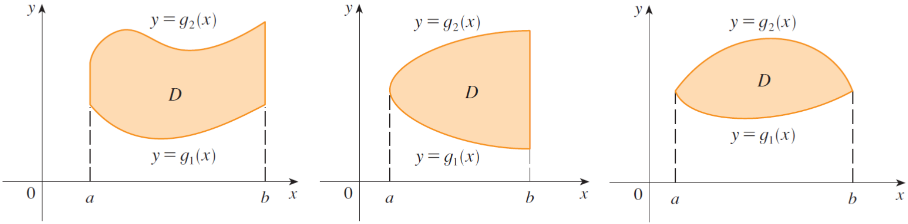
$$= \int_0^2 (2x - 26) dx = (x^2 - 26x)\Big|_{x=0}^{x=2} = -48.$$

Double integrals over general regions: type I regions

A region $D \subset \mathbb{R}^2$ is said to be of type I if it lies between the graphs of two continuous functions of x, that is:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

where g_1, g_2 are continuous and $g_1(x) \leq g_2(x)$ for every $x \in [a, b]$.



For a continuous function $f : D \to \mathbb{R}^1$ we have:

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Example: double integral over a type I region

Considering the function $f(x,y) = x + 2y$ defined on the type I region D bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$, we can write:

$$D = \{(x,y) \in \mathbb{R}^2 \ : \ -1 \le x \le 1 \text{ and } 2x^2 \le y \le 1 + x^2\}$$

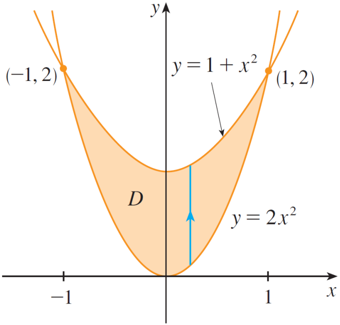
$$\iint_D (x+2y)dxdy =$$

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)dy\, dx =$$

$$= \int_{-1}^{1} (xy + y^2)\Big|_{y=2x^2}^{y=1+x^2} dx =$$

$$= \int_{-1}^{1} \left[x(1-x^2) + (1+x^2)^2 - (2x^2)^2\right] dx =$$

$$= \int_{-1}^{1} (1 + x + 2x^2 - x^3 - 3x^4)dx = \frac{32}{15}$$



Double integrals over general regions: type II regions

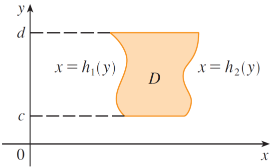
A region $D \subset \mathbb{R}^2$ is said to be of type II if it can be expressed as:

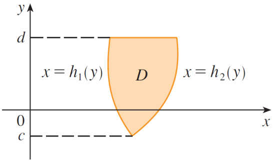
$D = \{(x, y) \in \mathbb{R}^2 \; : \; c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)\}$

where h_1, h_2 are continuous and $h_1(y) \leq h_2(y)$ for every $y \in [c, d]$.

For a continuous function $f : D \to \mathbb{R}^1$ we have:

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$



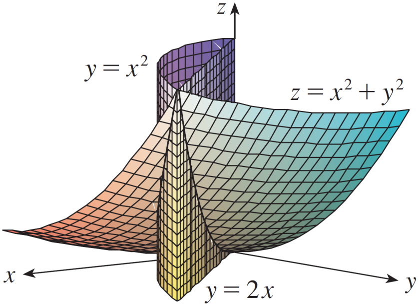


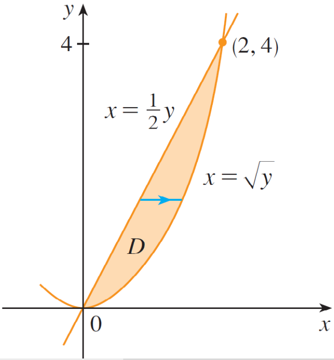
Example: double integral over a type II region

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq 2 \text{ and } x^2 \leq y \leq 2x\} \quad \text{or}$$

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq y \leq 4 \text{ and } \frac{1}{2}y \leq x \leq \sqrt{y}\}$$





Example: double integral over a type II region

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

We chose to express the region D as a type II region:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \le y \le 4 \text{ and } \frac{1}{2}y \le x \le \sqrt{y}\}$$

The volume can be computed as

$$V = \iint_D f(x, y)dxdy = \iint_D (x^2 + y^2)dxdy = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)dx\, dy =$$

$$= \int_0^4 \left(\frac{1}{3}x^3 + xy^2\right)\Big|_{x=y/2}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} - \frac{y^3}{24} + y^{5/2} - \frac{y^3}{2}\right) dy = \frac{216}{35}$$

Change of variables in double integrals

Theorem

If $A, B \subset \mathbb{R}^2$ are Jordan measurable sets, $T : B \to A$ is a bijection such that T and T^{-1} have continuous partial derivatives and $f : A \to \mathbb{R}^1$ is an integrable function, then the following equality holds:

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(x(\xi,\eta), y(\xi,\eta)) \left| \begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{array} \right| d\xi\,d\eta$$

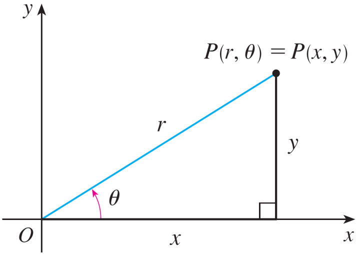
In the above theorem, the changes of variables are:

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$$

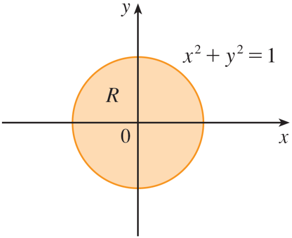
Double integrals in polar coordinates

The polar coordinates (r, θ) of a point P of the $\mathbb{R}^2$ plane are related to the rectangular (cartesian) coordinates (x, y) as:

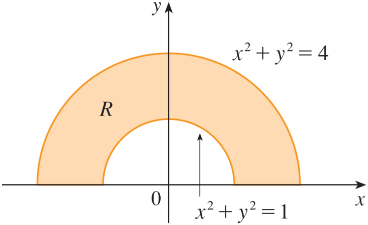
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad r \geq 0,\ \theta \in [0, 2\pi]$$



Examples:



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Double integrals in polar coordinates

Change of variables to polar coordinates in a double integral:

With the change of variables

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad , \ (r,\theta) \in R$$

we can compute:

$$\iint\limits_{D} f(x,y)\,dx\,dy = \iint\limits_{R} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$$

where D is the region for cartesian coordinates and R is the corresponding region for the polar coordinates.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \ !$$

Double integrals in polar coordinates: example

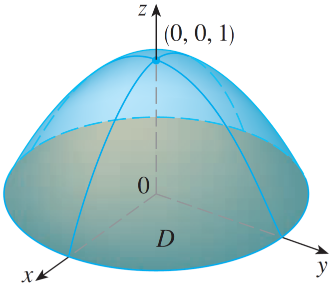
Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Intersection of paraboloid with xy-plane:

$$x^2 + y^2 = 1.$$

The solid lies above the disk:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$$



Region in polar coordinates - rectangle:

$$R = \{(r, \theta) : r \in [0, 1], \ \theta \in [0, 2\pi]\}$$

$$V = \iint_D f(x, y) dxdy = \iint_D (1 - x^2 - y^2) dxdy = \iint_R (1 - r^2) r \ dr \ d\theta =$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3) dr \ d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_{r=0}^{r=1} d\theta = 2\pi \frac{1}{4} = \frac{\pi}{2}.$$

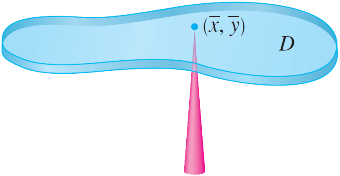
Applications of double integrals

- computing volumes: $V = \iint_D f(x,y)dxdy$
- density and mass: the mass of a lamina occupying the region D and having density function $\rho(x,y)$ is

$$m = \iint_D \rho(x,y)dxdy$$

- center of mass: the coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)dxdy \qquad \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)dxdy$$



Applications of double integrals

- computing surface areas: the area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous is:

$$A(S) = \iint_D \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}\, dx\, dy$$

Example: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

$D = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 9\} \quad \longrightarrow \quad R = [0, 3]\times[0, 2\pi]$ for polar coordinates

$$\begin{aligned} A(S) &= \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dx\, dy = \\ &= \iint_D \sqrt{4(x^2 + y^2) + 1}\, dx\, dy \\ &= \iint_R \sqrt{4r^2 + 1} \cdot r\, dr\, d\theta = \\ &= 2\pi \frac{1}{8}\frac{2}{3}(4r^2 + 1)^{3/2}\Big|_{r=0}^{r=3} = \frac{\pi}{6}(37\sqrt{37} - 1) \end{aligned}$$