CALCULUS HANDOUT 9 - PARTIAL AND DIRECTIONAL DERIVATIVES. DIFFERENTIABILITY. FRÉCHET DERIVATIVE - definitions

PARTIAL DERIVATIVES

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued function of n variables and $a = (a_1, a_2, ..., a_n) \in \text{Int}(A)$. The function f is said to be **partially differentiable with respect to x_i at** a if the following limit exists and is finite

$$\lim_{t\to 0} \frac{f(a_1, ..., a_{i-1}, a_i + t, a_{i+1}, ..., a_n) - f(a_1, ..., a_{i-1}, a_i, a_{i+1}, ..., a_n)}{t}$$

The value of this limit is denoted by $\frac{\partial f}{\partial x_i}(a)$ and is called the **partial derivative of f with respect to x_i at** a.

The vector $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), ..., \frac{\partial f}{\partial x_n}(a)\right)$ is called **gradient vector** of f at a.

! To calculate partial derivatives, one has to differentiate (in the normal manner) with respect to x_i keeping all the other variables fixed.
! All obvious rules for partially differentiating sums, products and quotients can be used.
! The partial differentiability of a vector valued function of n real variables is equivalent to the partial differentiability of all the scalar components.

DIRECTIONAL DERIVATIVES

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued function of n variables and $a \in \text{Int}(A)$ and $u \in \mathbb{R}^n$ s.t. $\|u\| = 1$.
If the following limit exists and is finite

$$\lim_{t\to 0} \frac{f(a + t \cdot u) - f(a)}{t}$$

it is called the **directional derivative in the direction** u of f at the point a and it is denoted by $\nabla_u f(a)$.

! If $e_i = (0, ..., 0, \underbrace{1}_{i}, 0, ..., 0)$ then $\nabla_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a)$, $i = \overline{1, n}$.
! Partial derivatives are special cases of directional derivatives.
! Relationship between directional derivative and gradient vector: $\nabla_u f(a) = \nabla f(a) \cdot u$ (where $\|u\| = 1$)

Theorem. Let f be a real valued function of n variables, $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$, and $a \in \text{Int}(A)$. If the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = \overline{1, n}$ exist in a neighborhood of a and they are continuous at a, then the following equality holds:

$$\lim_{h\to 0} \frac{f(a + h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0.$$

DIFFERENTIABILITY. FRÉCHET DERIVATIVE.

A <u>real valued</u> function of n variables $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ is said to be **differentiable** at a if it is partially differentiable at a with respect to every variable x_i and

$$\lim_{h\to 0} \frac{f(a + h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0.$$

Fréchet derivative of f at a: the function $d_a f : \mathbb{R}^n \to \mathbb{R}^1$ defined by

$$d_a f(h) = \nabla f(a) \cdot h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot h_i$$

! The Fréchet derivative $d_a f : \mathbb{R}^n \to \mathbb{R}^1$ is a linear function on $\mathbb{R}^n$. It is a polynomial of first degree in $h_1, h_2, ..., h_n$.
! For $\|h\| = 1$, we have $d_a(h) = \nabla_h f(a)$.
! If the function $f :\subset \mathbb{R}^n \to \mathbb{R}^1$ is differentiable at $a \in A$, then it is continuous at a.

A <u>vector valued</u> function of n variables $f = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $a \in \text{Int}(A)$ if every scalar component $f_j, j = \overline{1, m}$ of f is differentiable at a.
The **Fréchet derivative of f at** a is the function $d_a f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$d_a f(h) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(a) \cdot h_i\right) \cdot e_j \qquad \text{where } e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m.$$

The matrix of the linear function $d_a f$ is called the **Jacobi matrix** of f at a: $J_a(f) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{m\times n}$

We have $d_a f(h) = J_a(f) \cdot h$.

Composite rule:
Let $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}^p$. If f is differentiable at $a \in \text{Int}(A)$ and g is differentiable at $f(a) = b \in \text{Int}(B)$, then $h = g \circ f$ is differentiable at a and $d_a h = d_b g \circ d_a f$.

The Jacobi matrix of h at a is the product of the Jacobi matrix of g at b and the Jacobi matrix of f at a:

$$\frac{\partial h_i}{\partial x_j}(a) = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(b) \cdot \frac{\partial f_k}{\partial x_j}(a), \quad i = \overline{1,p}, j = \overline{1,n}.$$

Inverse rule:
Let $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ be a bijection where A, B are open subsets of $\mathbb{R}^n$. If f is differentiable at $a \in A$ and f^{-1} is differentiable at $b = f(a)$, then $d_a f$ is a bijection of $\mathbb{R}^n$ on $\mathbb{R}^n$ and $(d_a f)^{-1} = d_{f(a)} f^{-1}$.

Continuously differentiable functions:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function on the open set A. If the partial derivatives $A \ni x \mapsto \dfrac{\partial f_i}{\partial x_j}$ are continuous, $i = \overline{1,m}$, $j = \overline{1,n}$, then f is said to be **continuously differentiable**.

CALCULUS HANDOUT 9 - PARTIAL AND DIRECTIONAL DERIVATIVES. DIFFERENTIABILITY. FRÉCHET DERIVATIVE - examples

Ex.1 Compute the first order partial derivatives and the Fréchet derivative for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 3x^2y + e^{xy} \cdot \ln(x^2+y^2)$.

Solution:

Partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 3y \cdot (x^2)'_x + (e^{xy})'_x \cdot \ln(x^2+y^2) + e^{xy} \cdot \big(\ln(x^2+y^2)\big)'_x \\ &= 3y \cdot 2x + e^{xy} \cdot (xy)'_x \cdot \ln(x^2+y^2) + e^{xy} \cdot \frac{(x^2+y^2)'_x}{x^2+y^2} \\ &= 6xy + ye^{xy}\ln(x^2+y^2) + \frac{2x}{x^2+y^2} \cdot e^{xy}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= 3x^2 \cdot (y)'_y + (e^{xy})'_y \cdot \ln(x^2+y^2) + e^{xy} \cdot \big(\ln(x^2+y^2)\big)'_y \\ &= 3x^2 \cdot 1 + e^{xy} \cdot (xy)'_y \cdot \ln(x^2+y^2) + e^{xy} \cdot \frac{(x^2+y^2)'_y}{x^2+y^2} \\ &= 3x^2 + xe^{xy}\ln(x^2+y^2) + \frac{2y}{x^2+y^2} \cdot e^{xy}\end{aligned}$$

Fréchet derivative:

$$d_{(a_1,a_2)}f(h_1,h_2) = \frac{\partial f}{\partial x}(a_1,a_2) \cdot h_1 + \frac{\partial f}{\partial y}(a_1,a_2) \cdot h_2$$

$$= \left(6a_1a_2 + a_2e^{a_1a_2}\ln(a_1^2+a_2^2) + \frac{2a_1}{a_1^2+a_2^2} \cdot e^{a_1a_2}\right)h_1 + \left(3a_1^2 + a_1e^{a_1a_2}\ln(a_1^2+a_2^2) + \frac{2a_2}{a_1^2+a_2^2} \cdot e^{a_1a_2}\right)h_2$$

$$= 3a_1(2a_2h_1 + a_1h_2) + e^{a_1a_2}\ln(a_1^2+a_2^2)(a_2h_1+a_1h_2) + \frac{2e^{a_1a_2}}{a_1^2+a_2^2}(a_1h_1+a_2h_2)$$

Ex.2 Compute the directional derivative in the direction $u = \dfrac{v}{\|v\|}$, where $v = (1,1)$ at $a = (2,1)$ for the function

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = x^2 + 2xy + 3y^2.$$

Solution:

Method I: Using the definition $\nabla_u f(a) = \lim\limits_{t\to 0} \dfrac{f(a+t\cdot u) - f(a)}{t}$.

$\|v\| = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow u = \dfrac{v}{\|v\|} = \dfrac{1}{\sqrt{2}}(1,1) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

$$\begin{aligned}f(a+t\cdot u) &= f\left((2,1) + t\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right) = f\left(2+\frac{\sqrt{2}}{2}t, 1+\frac{\sqrt{2}}{2}t\right) \\ &= \left(2+\frac{\sqrt{2}}{2}t\right)^2 + 2\left(2+\frac{\sqrt{2}}{2}t\right)\left(1+\frac{\sqrt{2}}{2}t\right) + 3\left(1+\frac{\sqrt{2}}{2}t\right)^2 \\ &= 4 + 2\cdot 2\cdot\frac{\sqrt{2}}{2}t + \frac{2}{4}t^2 + 2\left(2 + 2\cdot\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t + \frac{2}{4}t^2\right) + 3\left(1 + 2\cdot\frac{\sqrt{2}}{2}t + \frac{2}{4}t^2\right) \\ &= 4 + 2\sqrt{2}t + \frac{1}{2}t^2 + 4 + 2\sqrt{2}t + \sqrt{2}t + t^2 + 3 + 3\sqrt{2}t + \frac{3}{2}t^2 \\ &= 11 + 8\sqrt{2}t + 3t^2\end{aligned}$$

$f(a) = f(2,1) = 2^2 + 2\cdot 2\cdot 1 + 3\cdot 1^2 = 4+4+3 = 11$

$\Rightarrow \nabla_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} f(2,1) = \lim\limits_{t\to 0}\dfrac{11 + 8\sqrt{2}t + 3t^2 - 11}{t} = \lim\limits_{t\to 0}\dfrac{t(8\sqrt{2}+3t)}{t} = \lim\limits_{t\to 0}(8\sqrt{2}+3t) = 8\sqrt{2}.$

Method II: Using the formula $\nabla_u f(a) = \nabla f(a) \cdot u$.

$\dfrac{\partial f}{\partial x}(x,y) = \big(x^2+2xy+3y^2\big)'_x = (x^2)'_x + 2y\cdot(x)'_x + (3y^2)'_x = 2x + 2y\cdot 1 + 0 = 2x+2y$

$\dfrac{\partial f}{\partial y}(x,y) = \big(x^2+2xy+3y^2\big)'_y = (x^2)'_y + 2x\cdot(y)'_y + 3(y^2)'_y = 0 + 2x\cdot 1 + 3\cdot 2y = 2x+6y$

$\Rightarrow \dfrac{\partial f}{\partial x}(2,1) = 2\cdot 2 + 2\cdot 1 = 4+2 = 6$ and $\dfrac{\partial f}{\partial y}(2,1) = 2\cdot 2 + 6\cdot 1 = 4+6 = 10$

$\Rightarrow \nabla_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} f(2,1) = (6,10)\cdot\left(\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right) = 6\cdot\dfrac{\sqrt{2}}{2} + 10\cdot\dfrac{\sqrt{2}}{2} = \dfrac{6\sqrt{2}+10\sqrt{2}}{2} = \dfrac{16\sqrt{2}}{2} = 8\sqrt{2}.$

Ex.3 Study wether the following function is continuous, partially differentiable, (Fréchet) differentiable or continuously differentiable on $\mathbb{R}^2$:

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}.$$

Solution:

Continuity:

The function f is continuous on $\mathbb{R}^2\setminus\{(0,0)\}$. Moreover, $f(0,0)=0$.

We study if the function f has limit at the point $(0,0)$.

$y = x \Rightarrow f(x,x) = \dfrac{x\cdot x}{x^2+y^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2} \xrightarrow[x\to 0]{} \dfrac{1}{2}$

$y = -x \Rightarrow f(x,-x) = \dfrac{x\cdot(-x)}{x^2+(-x)^2} = \dfrac{-x^2}{2x^2} = -\dfrac{1}{2} \xrightarrow[x\to 0]{} -\dfrac{1}{2}$

As $\dfrac{1}{2}\neq -\dfrac{1}{2}$, it results that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

We obtain that f is not continuous at the point $(0,0)$.

Therefore, the function f is continuous on $\mathbb{R}^2\setminus\{(0,0)\}$

Partial derivatives:

$\dfrac{\partial f}{\partial x}(x,y) = \lim\limits_{t\to 0}\dfrac{f(x+t,y)-f(x,y)}{t} \Rightarrow \dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{t\to 0}\dfrac{f(0+t,0)-f(0,0)}{t} = \lim\limits_{t\to 0}\dfrac{f(t,0)-0}{t} = 0$

$\dfrac{\partial f}{\partial y}(x,y) = \lim\limits_{t\to 0}\dfrac{f(x,y+t)-f(x,y)}{t} \Rightarrow \dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{t\to 0}\dfrac{f(0,0+t)-f(0,0)}{t} = \lim\limits_{t\to 0}\dfrac{f(0,t)-0}{t} = 0$

It follows that f is partially differentiable at $(0,0)$. Moreover, the partial derivatives of the function are:

$\dfrac{\partial f}{\partial x}(x,y) = \dfrac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2} = \dfrac{x^2y+y^3-2x^2y}{(x^2+y^2)^2} = \dfrac{y^3-x^2y}{(x^2+y^2)^2}$

$\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x(x^2+y^2)-xy\cdot 2y}{(x^2+y^2)^2} = \dfrac{x^3+xy^2-2xy^2}{(x^2+y^2)^2} = \dfrac{x^3-xy^2}{(x^2+y^2)^2}$

Fréchet differentiability:

At the point $(0,0)$ we have that $f(0,0)=0$, $\dfrac{\partial f}{\partial x}(0,0)=0$ and $\dfrac{\partial f}{\partial y}(0,0)=0$.

We check if $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)-\nabla f(a)\cdot h}{\|h\|} = 0$, where $a=(0,0)$ and $h=(h_1,h_2)$.

We have that $\nabla f(0,0) = \left(\dfrac{\partial f}{\partial x}(0,0),\dfrac{\partial f}{\partial y}(0,0)\right) = (0,0)$ and $\|h\| = \|(h_1,h_2)\| = \sqrt{h_1^2+h_2^2}$. Then:

$\lim\limits_{h\to 0}\dfrac{f\left((0,0)+(h_1,h_2)\right)-f(0,0)-(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim\limits_{h\to 0}\dfrac{f(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim\limits_{h\to 0}\dfrac{h_1h_2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}.$

The limit does not exsit for $(h_1,h_2)\to(0,0)$. (check!)

It results that the functions f is not Fréchet differentiable at the point $(0,0)$.

Continuous differentiability:

$\dfrac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^3-x^2y}{(x^2+y^2)^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$ and $\dfrac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3-xy^2}{(x^2+y^2)^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

We have that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are not continuous at the point $(0,0)$. (check!)

Thus, the function f is not continuously differentiable on $\mathbb{R}^2$.

CALCULUS HANDOUT 9 - PARTIAL AND DIRECTIONAL DERIVATIVES. DIFFERENTIABILITY. FRÉCHET DERIVATIVE - exercises

1. Compute the first order partial derivatives and the Fréchet derivative for the following functions:

   1. $f(x,y) = 3x^2 - 4xy + 5y^2$
   2. $f(x,y) = e^{-xy}$
   3. $f(x,y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$
   4. $f(x,y) = e^x(\cos y - \sin y)$
   5. $f(x,y) = \arctan xy$
   6. $f(x,y) = x^4 + 5xy^3$
   7. $f(x,y) = y^2e^{-x}$
   8. $f(x,y) = \dfrac{x}{y}$
   9. $f(x,y) = \dfrac{ax+by}{cx+dy}$
   10. $f(x,y) = (x^2y - y^3)^5$
   11. $f(x,y) = x^2y - 3y^4$
   12. $f(x,y) = \sqrt{3x+4y}$
   13. $f(x,y) = x\sin(xy)$
   14. $f(x,y) = \dfrac{x}{(x+y)^2}$
   15. $f(x,y) = \dfrac{e^y}{x+y^2}$
   16. $f(x,y) = \arctan(xy^2)$
   17. $f(x,y) = \ln(x^2+y^2)$
   18. $f(x,y) = x^y$
   19. $f(x,y,z) = x^2e^y\ln z$
   20. $f(x,y,z) = e^{xyz}$
   21. $f(x,y,z) = xe^y + ye^z + ze^x$
   22. $f(x,y,z) = x^3yz^2 + 2yz$
   23. $f(x,y,z) = \ln(x+2y+3z)$
   24. $f(x,y,z) = \sqrt{x^4 + y^2\cos z}$
   25. $f(x,y,z) = x^2y\cos\dfrac{z}{x}$
   26. $f(x,y,z) = xy^2e^{-xz}$
   27. $f(x,y,z) = y\tan(x+2z)$
   28. $f(x,y,z) = (x^2+y^2)\sin(x+2y+z^2)$

2. Find the directional derivative of the following functions at the given point a in the direction $u = \dfrac{v}{\|v\|}$:

   1. $f(x,y) = 2x^2 + 3xy^2 + y^2$; $a = (1,-1)$, $v = (3,4)$
   2. $f(x,y) = e^x\sin y$; $a = (0,\pi/4)$, $v = (1,-1)$
   3. $f(x,y) = x^3 - x^2y + xy^2 + y^3$; $a = (1,-1)$, $v = (2,3)$
   4. $f(x,y,z) = xy + yz + zx$; $a = (1,-1,2)$, $v = (1,2,-2)$
   5. $f(x,y,z) = \ln(1+x^2+y^2-z^2)$; $a = (1,-1,1)$; $v = (2,-2,-3)$

3. Study wether the following functions are continuous, partially differentiable, (Fréchet) differentiable or continuously differentiable on $\mathbb{R}^2$:

   1. $f(x,y) = (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   2. $f(x,y) = \dfrac{e^{xy}-1}{x}$ if $x \neq 0$ and $f(0,y) = y$
   3. $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   4. $f(x,y) = \dfrac{4xy(x^2-y^2)}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   5. $f(x,y) = \dfrac{\ln(1+xy)}{x}$ if $x \neq 0$ and $f(0,y) = y$
   6. $f(x,y) = (x^2+y^2)\sin\dfrac{1}{xy}$ if $xy \neq 0$ and $f(x,y) = 0$ if $xy = 0$
   7. $f(x,y) = \dfrac{x^2(1-\cos(xy))}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   8. $f(x,y) = \dfrac{x^3}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   9. $f(x,y) = \dfrac{x^ay}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$ $(a \geq 1)$
   10. $f(x,y) = y^2\cos\dfrac{1}{x}$ if $x \neq 0$ and $f(0,y) = 0$
   11. $f(x,y) = \dfrac{\sin(x^4)}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$
   12. $f(x,y) = \dfrac{\mathrm{tg}(x^3y)}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$

Extra exercises

4. Find the partial derivatives of the function $z = z(x, y)$ given implicitly by the following equations:
   1. $x^{2/3} + y^{2/3} + z^{2/3} = 1$
   2. $x^3 + y^3 + z^3 = xyz$
   3. $xe^{yz} + ye^{zx} + ze^{xy} = 3$
   4. $\sin(xy) + \sin(yz) + \sin(zx) = 1$
   5. $x^5 + xy^2 + yz = 5$
   6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
   7. $xyz = \sin(x + y + z)$

5. Find a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = 2xy^3 + e^x \sin y$ and $\frac{\partial f}{\partial y} = 3x^2y^2 + e^x \cos y + 1$.