CALCULUS HANDOUT 8
FUNCTIONS OF SEVERAL VARIABLES. LIMITS AND CONTINUITY - definitions

THE VECTOR SPACE $\mathbb{R}^n$

$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}^1, i = 1, 2, ..., n\}$. The elements of $\mathbb{R}^n$ are called **vectors**.

$\mathbb{R}^n$ is a n-**dimensional vector space** with respect to the sum and the scalar product defined by:
$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
$k(x_1, x_2, \dots, x_n) = (kx_1, kx_2, \dots, kx_n)$

For $x \in \mathbb{R}^n$ **the norm** (or length) of x is defined by $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

The distance between x and $a = (a_1, a_2, \dots, a_n)$ is $\|x - a\|$.

A **neighborhood** of $a \in \mathbb{R}^n$ is a set $V \subset \mathbb{R}^n$ which contains a hypersphere $S_r(a)$ centered in a, $S_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$, $r > 0$.

FUNCTIONS OF SEVERAL VARIABLES

A **real valued function of** n **variables** associates to every vector $x \in A \subset \mathbb{R}^n$ a unique real number. Formally, $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ is given by $x = (x_1, x_2, \dots, x_n) \in A \mapsto f(x) = f(x_1, x_2, \dots, x_n) \in \mathbb{R}$.

A **vector valued function of** n **variables** associates to every vector $x \in A \subset \mathbb{R}^n$ a unique vector $f(x)$ from $\mathbb{R}^m$. Formally, $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is given by
$x = (x_1, x_2, ..., x_n) \in A \mapsto f(x) = (f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_m(x_1, x_2, \dots, x_n)) \in \mathbb{R}^m$
The functions $f_i : A \subset \mathbb{R}^n \to \mathbb{R}^1$, $i = \overline{1, m}$, are called **scalar components** of the vector function f.

SEQUENCES IN $\mathbb{R}^n$

A **sequence** (x_k) of vectors of $\mathbb{R}^n$ is a function whose domain is $\mathbb{N}$ and whose values belong to $\mathbb{R}^n$.

A vector $x \in \mathbb{R}^n$ is said to be **the limit of the sequence** (x_k) if for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for any $k > N$ we have $\|x_k - x\| < \varepsilon$. In this case we write $\lim_{k\to\infty} x_k = x$.

Properties:
- If the limit of the sequence (x_k) exists, then it is unique.
- If a sequence (x_k) converges to x, then the sequence is bounded: $\exists\, M > 0\, s.t.\, \|x_k\| < M$, $\forall k \in \mathbb{N}$.
- If a sequence (x_k) converges to x, then any subsequence (x_{k_l}) of the sequence (x_k) converges to x.
- A sequence (x_k), $x_k = (x_{1k}, x_{2k}, ..., x_{nk}) \in \mathbb{R}^n$ converges to $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ if and only if the sequence (x_{ik}) converges to x_i for any $i = 1, 2, ..., n$.
- **Bolzano-Weierstrass**: any bounded sequence (x_k) of points of $\mathbb{R}^n$ contains a convergent subsequence.
- **Cauchy's criterion for convergence**: A sequence $(x_k) \subset \mathbb{R}^n$ converges if and only if for any $\varepsilon > 0$ there exists N_ε such that for $p, q > N_\varepsilon$ we have $\|x_p - x_q\| < \varepsilon$.

LIMITS

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ be a real valued n variable function and $a \in A'$ (i.e., for any neighborhood V of a, one has $V \setminus \{a\} \cap A \neq \emptyset$). The real number L is called the **limit** of $f(x)$ as x tends to a if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $0 < \|x - a\| < \delta$ then $|f(x) - L| < \varepsilon$. We write $\lim_{x\to a} f(x) = L$.

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function of n variables and $a \in A'$. The vector $L \in \mathbb{R}^m$ is called the **limit** of $f(x)$ as x tends to a, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \|x - a\| < \delta$ then $\|f(x) - L\| < \varepsilon$. We write $L = \lim_{x\to a} f(x)$.

! If $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ and $L = (L_1, \dots, L_m)$, then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a} f_i(x) = L_i$ for any $i = \overline{1, m}$.

! same limit laws as for functions of one real variable **!**

Heine's criterion for the limit:
The function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ has a limit as x approaches a if and only if for any sequence (x_k), $x_k \in A$, $x_k \neq a$, and $x_k \to a$ as $k \to \infty$, the sequence $(f(x_k))$ converges.

Cauchy-Bolzano's criterion for the limit:
The function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ has a limit as $x \to a$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \|x' - a\| < \delta$ and $0 < \|x'' - a\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

CONTINUITY

A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$.

Rules for continuity:

• If the real valued functions of n variables f and g are continuous at a then so are $f + g$, $f \cdot g$ and $\frac{1}{f}$.

• If $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ is continuous at $a \in A$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}^p$ is continuous at $f(a) = b \in \mathbb{R}^m$, then the composite function $g \circ f : A \to \mathbb{R}^p$ is continuous at a.

Continuity of the scalar components:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = (f_1(x), \dots, f_m(x))$ and $a \in A$. The function f is continuous at $a \in A$ if and only if the scalar components f_i, $i = 1, 2, \dots, m$ are continuous at a.

Heine's criterion for continuity:
The function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in A$ if and only if for any sequence $(x_k) \subset A$ which converges to a, the sequence $(f(x_k))$ converges to $f(a)$.

Cauchy-Bolzano's criterion for continuity:
The function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in A$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x' - a\| < \delta$ and $\|x'' - a\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

The boundedness property:
If $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on the compact set A, then the set $f(A)$ is bounded and there exists $a \in A$ such that $\|f(a)\| = \sup \|f(A)\|$.

Uniform continuity:
A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$ is **uniformly continuous** (on A) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $x', x'' \in A$ we have that if $\|x' - x''\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

Ex.1 We study if the following functions have limits at the given points::

a. $f(x,y) = 5x^2 - 2xy + y^2 - 6$ at $(1,2)$

b. $f(x,y) = \dfrac{x^2 - y^2}{x+y}$ at $(0,0)$

c. $f(x,y) = (x^2+y^2)\cos\dfrac{1}{x+y}$ at $(0,0)$

d. $f(x,y) = \dfrac{xy}{x^2+y^2}$ at $(0,0)$

Solution:

a. $f(x,y) = 5x^2 - 2xy + y^2 - 6$ at $(1,2)$

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} (5x^2 - 2xy + y^2 - 6) = 5\cdot 1^2 - 2\cdot 1\cdot 2 + 2^2 - 6 = 5 - 4 + 4 - 6 = -1$$

b. $f(x,y) = \dfrac{x^2 - y^2}{x+y}$ at $(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} (x-y) = 0 - 0 = 0$$

c. $f(x,y) = (x^2+y^2)\cos\dfrac{1}{x+y}$ în $(0,0)$

$$-1 \le \cos\frac{1}{x+y} \le 1 \Big| \cdot (x^2+y^2) \ge 0 \Leftrightarrow -(x^2+y^2) \le (x^2+y^2)\cos\frac{1}{x+y} \le x^2+y^2$$

$$\lim_{(x,y)\to(0,0)} -(x^2+y^2) = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0^2 + 0^2 = 0$$

Applying the squeeze rule, it follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

d. $f(x,y) = \dfrac{xy}{x^2+y^2}$ în $(0,0)$

We compute the limits of the functions along the lines $y=0$ and $y=x$.

$$y = 0 \Rightarrow f(x,0) = \frac{x\cdot 0}{x^2} = 0 \xrightarrow[x\to 0]{} 0$$

$$y = x \Rightarrow f(x,x) = \frac{x\cdot x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2} \xrightarrow[x\to 0]{} \frac{1}{2}$$

As we obtained two different limits along two different lines, it results that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

Ex.2 We study the continuity of the function $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \arcsin\dfrac{y^2}{(x-1)^2+y^2} & ,(x,y) \ne (1,0) \\ 0 & ,(x,y) = (1,0) \end{cases}$.

Solution: We study the continuity of the function f at the point $(1,0)$.

As $f(1,0) = 0$, we compute, if it exists, $\lim\limits_{(x,y)\to(1,0)} f(x,y)$.

Consider the sequences $\left(1, \dfrac{1}{n}\right) \xrightarrow[n\to\infty]{} (1,0)$ and $\left(1+\dfrac{1}{n}, \dfrac{1}{n}\right) \xrightarrow[n\to\infty]{} (1,0)$.

Compute $f\left(1,\dfrac{1}{n}\right)$ and $f\left(1+\dfrac{1}{n},\dfrac{1}{n}\right)$.

$$f\left(1,\frac{1}{n}\right) = \arcsin\frac{\frac{1}{n^2}}{(1-1)^2 + \frac{1}{n^2}} = \arcsin 1 = \frac{\pi}{2} \xrightarrow[n\to\infty]{} \frac{\pi}{2}$$

$$f\left(1+\frac{1}{n},\frac{1}{n}\right) = \arcsin\frac{\frac{1}{n^2}}{\left(1+\frac{1}{n}-1\right)^2 + \frac{1}{n^2}} = \arcsin\frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \arcsin\frac{1}{2} = \frac{\pi}{6} \xrightarrow[n\to\infty]{} \frac{\pi}{6}$$

As $\dfrac{\pi}{2} \ne \dfrac{\pi}{6}$, applying Heine's theorem, it follows that $\lim\limits_{(x,y)\to(1,0)} f(x,y)$ doesn't exist.

Thus, function f is not continuous at $(1,0)$.

CALCULUS HANDOUT 8
FUNCTIONS OF SEVERAL VARIABLES. LIMITS AND CONTINUITY - exercises

1. Study if the following functions have limits at the given points:

1. $f(x,y) = 3x^2 - 4xy + 5y^2$ at $(1,-2)$
2. $f(x,y) = x^2y^3 - 4y^2$ at $(3,2)$
3. $f(x,y) = e^{\sqrt{2x-y}}$ at $(3,2)$
4. $f(x,y) = e^{-xy}$ at $(1,-1)$
5. $f(x,y) = \dfrac{x+y}{1+xy}$ at $(0,0)$
6. $f(x,y) = \dfrac{x^2y+xy^2}{x^2-y^2}$ at $(2,-1)$
7. $f(x,y) = \dfrac{\cos(x^2+y^2)}{1-x^2-y^2}$ at $(0,0)$
8. $f(x,y,z) = \sqrt{xy}\tan\dfrac{3\pi z}{4}$ at $(2,8,1)$
9. $f(x,y) = \dfrac{4-xy}{4+xy}$ at $(2,-2)$
10. $f(x,y) = \dfrac{\mathrm{ctg}(x^2+y^2)}{x^2+y^2}$ at $(0,0)$
11. $f(x,y) = \dfrac{\sin(xy)}{x}$ at $(0,0)$ and $(0,2)$
12. $f(x,y) = \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$ at $(0,0)$
13. $f(x,y) = (x^2+y^2)\sin\dfrac{1}{xy}$ at $(0,0)$
14. $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ at $(0,0)$
15. $f(x,y) = \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$ at $(0,0)$
16. $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ at $(0,0)$
17. $f(x,y) = \dfrac{x^4-4y^2}{x^2+2y^2}$ at $(0,0)$
18. $f(x,y) = \dfrac{y^2\sin^2 x}{x^4+y^4}$ at $(0,0)$
19. $f(x,y) = \dfrac{x^4+y^4}{(x^2+y^2)^{3/2}}$ at $(0,0)$
20. $f(x,y) = \dfrac{x^2+y^2}{|x|+|y|}$ at $(0,0)$
21. $f(x,y) = \dfrac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}$ at $(0,0)$
22. $f(x,y) = \dfrac{2x^2-3y^2}{x^2+y^2}$ at $(0,0)$
23. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ at $(0,0)$
24. $f(x,y) = \dfrac{x^4-y^4}{x^4+x^2y^2+y^4}$ at $(0,0)$
25. $f(x,y,z) = \dfrac{x^2+y^2-z^2}{x^2+y^2+z^2}$ at $(0,0,0)$
26. $f(x,y) = \dfrac{e^{xy}-1}{x^2+y^2}$ at $(0,0)$
27. $f(x,y) = \dfrac{xy^2}{x^2+y^6}$ at $(0,0)$
28. $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ at $(0,0)$
29. $f(x,y) = \dfrac{xy^2+\sin(x^3+y^5)}{x^2+y^4}$ at $(0,0)$
30. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ at $(0,0)$
31. $f(x,y) = \dfrac{xy^2\cos y}{x^2+y^4}$ at $(0,0)$
32. $f(x,y) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ at $(0,0)$
33. $f(x,y) = \dfrac{xy^4}{x^2+y^8}$ at $(0,0)$
34. $f(x,y) = \dfrac{xy-y}{(x-1)^2+y^2}$ at $(1,0)$
35. $f(x,y) = \dfrac{x^3-y^3}{x^2+xy+y^2}$ at $(0,0)$
36. $f(x,y) = \dfrac{xy^4}{x^4+y^4}$ at $(0,0)$

2. Discuss the continuity of the following functions:

1. $f(x,y) = (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$
2. $f(x,y) = \dfrac{\sin xy}{xy}$ if $xy\neq 0$ and $f(x,y)=1$ if $xy=0$
3. $f(x,y) = \dfrac{4xy(x^2-y^2)}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$
4. $f(x,y) = \dfrac{x^2(1-\cos(xy))}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$
5. $f(x,y) = \dfrac{x^2y^3}{2x^2+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$
6. $f(x,y) = \dfrac{xy}{x^2+xy+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$