CALCULUS HANDOUT 7 - FOURIER SERIES - definitions

FOURIER SERIES

The **Fourier series** of a piecewise continuous function f defined on the interval $[-\pi, \pi]$ is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cdot \cos nx + b_n \cdot \sin nx)$$

in which the **Fourier coefficients** a_n, b_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx \qquad \text{for } n = 0, 1, 2, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx \qquad \text{for } n = 1, 2, ...$$

Fourier Theorem:
Let f be a piecewise continuous function defined on the interval $[-\pi, \pi]$ and extended by periodicity outside it. We denote by $S_n(x)$ the n-th partial sum of the Fourier series defined above.

If $f(x)$ has finite left-hand and right-hand side derivatives at its points of discontinuity, then:
a) when $x = x_0$ is a point of continuity of f, then $\lim_{n\to\infty} S_n(x_0) = f(x_0)$.
b) when $x = x_0$ is a point of discontinuity of f, then $\lim_{n\to\infty} S_n(x_0) = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$.

Change of the origin of the fundamental interval:
If f is a piecewise continuous function defined on the fundamental interval $[\alpha - \pi, \alpha + \pi]$ and by periodic extension outside it, then for any α, the Fourier coefficients a_n, b_n are given by

$$a_n = \frac{1}{\pi}\int_{\alpha-\pi}^{\alpha+\pi} f(x) \cdot \cos nx \, dx \qquad \text{for } n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi}\int_{\alpha-\pi}^{\alpha+\pi} f(x) \cdot \sin nx \, dx \qquad \text{for } n = 1, 2, \dots$$

The Fourier series of $f(x)$ converges at every point of continuity and:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \text{for } x \in [\alpha - \pi, \alpha + \pi].$$

Change of the interval length:
If f is a piecewise continuous function defined on the interval $[-L, L]$ and by periodic extension outside it, then the Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \cos\frac{n\pi x}{L} \, dx \qquad \text{for } n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \sin\frac{n\pi x}{L} \, dx \qquad \text{for } n = 1, 2, \dots$$

The Fourier series (see below) of $f(x)$ converges at every point of continuity and:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

Odd and even functions:
If the function f is even, then $b_n = 0$ for any $n = 1, 2, \dots$.
If the function f is odd, then $a_n = 0$ for any $n = 0, 1, 2, \dots$.

Example: We determine the Fourier series for the function $f(x) = x^2$ for $x \in [-\pi, \pi]$.
Solution:
We can easily see that
$f(-x) = (-x)^2 = (-1)^2 \cdot x^2 = 1 \cdot x^2 = x^2 = f(x)$.
It results the the function f is even, and therefore $b_n = 0$, for any $n \geq 1$.
We compute a_0.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(0 \cdot x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot 1 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \cdot \frac{x^3}{3} \Big|_0^{\pi} = \frac{2}{\pi} \cdot \left(\frac{\pi^3}{3} - \frac{0^3}{3} \right) = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

We compute a_n, appyling the integration by parts method twice:

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx \\
&= \frac{2}{\pi} \int_0^{\pi} x^2 \left(\frac{sin(nx)}{n} \right)' dx = \frac{2}{\pi} \left(x^2 \cdot \frac{\sin(nx)}{n} \Big|_0^{\pi} - \int_0^{\pi} 2x \cdot \frac{\sin(nx)}{n} dx \right) \\
&= \frac{2}{\pi} \left(\pi^2 \cdot \frac{\sin(n\pi)}{n} - 0^2 \cdot \frac{\sin 0}{n} - \frac{2}{n} \int_0^{\pi} x \sin(nx) dx \right) \\
&= \frac{2}{\pi} \left(\pi^2 \cdot \frac{0}{n} - 0 - \frac{2}{n} \int_0^{\pi} x \sin(nx) dx \right) \\
&= -\frac{4}{n\pi} \int_0^{\pi} x \cdot \left(\frac{-\cos(nx)}{n} \right)' dx \\
&= -\frac{4}{n\pi} \left(-x \cdot \frac{\cos(nx)}{n} \Big|_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{-\cos(nx)}{n} dx \right) \\
&= -\frac{4}{n\pi} \left(-\pi \cdot \frac{\cos(n\pi)}{n} + 0 \cdot \frac{\cos 0}{n} + \frac{1}{n} \int_0^{\pi} \cos(nx) dx \right) \\
&= -\frac{4}{n\pi} \left(-\frac{\pi}{n} \cdot (-1)^n + 0 + \frac{1}{n} \cdot \frac{\sin(nx)}{n} \Big|_0^{\pi} \right) \\
&= -\frac{4}{n\pi} \left(-\frac{\pi}{n} \cdot (-1)^n + \frac{1}{n^2} \cdot \sin(n\pi) - \frac{1}{n^2} \cdot \sin 0 \right) \\
&= -\frac{4}{n\pi} \left(-\frac{\pi}{n} \cdot (-1)^n + 0 - 0 \right) \\
&= \frac{4 \cdot (-1)^n}{n^2}
\end{aligned}$$

The Fourier series associated to the function f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nx + b_n \cdot \sin nx) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos nx + 0 \cdot \sin nx) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot \cos nx.$$

Remark: For $x = \pi$, we obtain:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=2}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cos(n\pi) \Leftrightarrow 4 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2} \cdot (-1)^n = \pi^2 - \frac{\pi^2}{3}$$

$$\Leftrightarrow 4 \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3} \; \Big| : 4 \Leftrightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

CALCULUS HANDOUT 7 - FOURIER SERIES - exercises

1. Find the Fourier series for:
   1. $f(x) = \pi^2 - x^2$, $x \in [-\pi, \pi]$
   2. $f(x) = x^2$, $x \in [-\pi, \pi]$
   3. $f(x) = |x|$, $x \in [-\pi, \pi]$
   4. $f(x) = x^3, x \in [-1, 1]$
   5. $f(x) = \cos(ax)$, $x \in [-\pi, \pi]$
   6. $f(x) = \cosh(ax)$, $x \in [-\pi, \pi]$
   7. $f(x) = e^{ax}$, $x \in [-1, 1]$
   8. $f(x) = \sin(3\pi x)$, $x \in [-1, 1]$
   9. $f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ \cos x & \text{if } 0 \leq x \leq \pi \end{cases}$
   10. $f(x) = \begin{cases} a & \text{if } -\pi \leq x < 0 \\ b & \text{if } 0 \leq x \leq \pi \end{cases}$
   11. $f(x) = \begin{cases} -x & \text{if } -\frac{\pi}{2} \leq x \leq 0 \\ x & \text{if } 0 \leq x < \pi \\ 2\pi - x & \text{if } \pi \leq x \leq \frac{3\pi}{2} \end{cases}$
   12. $f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x \leq \pi \end{cases}$
   13. $f(x) = \begin{cases} -1 & \text{if } -4 \leq x < 0 \\ 3 & \text{if } 0 \leq x \leq 4 \end{cases}$

2. Find the Fourier series for:
   1. $f(x) = 1 - x$, $x \in [-1, 1]$, where $f(x + 2) = f(x)$
   2. $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } 1 \leq |x| \leq 2 \end{cases}$ where $f(x + 4) = f(x)$
   3. $f(x) = \begin{cases} -x & \text{if } -4 \leq x < 0 \\ 0 & \text{if } 0 \leq x \leq 4 \end{cases}$ where $f(x + 8) = f(x)$
   4. $f(x) = \begin{cases} 0 & \text{if } -2 \leq x < 0 \\ 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$ where $f(x + 4) = f(x)$