CALCULUS HANDOUT 6 - RIEMANN-DARBOUX INTEGRAL - definitions

THE RIEMANN DARBOUX INTEGRAL

A **partition** P of the interval $[a,b]$ is a finite set of points $\{x_0, x_1, .., x_n\}$ satisfying $a = x_0 < x_1 < .. < x_n = b$. Consider a function f defined and bounded on $[a,b]$.

Upper Darboux sum of f related to P: $U_f(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ where $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$.

Lower Darboux sum of f related to P: $L_f(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ where $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$.

Consider $M = \sup\{f(x) \,|\, a \le x \le b\}$ and $m = \inf\{f(x) \,|\, a \le x \le b\}$.
For any partition P of $[a,b]$ we have: $m(b-a) \le L_f(P) \le U_f(P) \le M(b-a)$.
$L_f = \{L_f(P) \,|\, P \text{ is a partition of } [a,b]\}$ and $U_f = \{U_f(P) \,|\, P \text{ is a partition of } [a,b]\}$ are bounded sets.
So $\mathcal{L}_f = \sup L_f$ and $\mathcal{U}_f = \inf U_f$ exist. Moreover, $\mathcal{L}_f \le \mathcal{U}_f$.

A function defined and bounded on $[a,b]$ is **Riemann-Darboux integrable** on $[a,b]$ if $\mathcal{L}_f = \mathcal{U}_f$.

This common value is denoted by $\int_a^b f(x)\,dx = \mathcal{L}_f = \mathcal{U}_f$.

Properties of the Riemann-Darboux integral:
If f and g are Riemann-Darboux integrable on $[a,b]$ then all the integrals below exist and

(1) $\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$ for any $\alpha, \beta \in \mathbb{R}$.

(2) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any $a \le c \le b$.

(3) If $f(x) \le g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

(4) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Classes of Riemann-Darboux integrable functions:
If f is *continuous* on $[a,b]$, then f is Riemann-Darboux integrable on $[a,b]$.

A function f is called *piecewise continuous* on $[a,b]$ if there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and continuous functions f_i defined on $[x_{i-1}, x_i]$, such that $f(x) = f_i(x)$ for $x \in (x_{i-1}, x_i)$, $i = 1, 2, \ldots, n$.

A piecewise continuous function is Riemann-Darboux integrable and $\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_i(x)\,dx$.

The integral mean value theorem:
If f and g are continuous on $[a,b]$ and $g(x) \ge 0$ for $x \in [a,b]$, then there exists c between a and b such that $\int_a^b f(x) \cdot g(x)\,dx = f(c) \int_a^b g(x)\,dx$.

The fundamental theorem of calculus:
If f is Riemann-Darboux integrable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$.

Furthermore, if f is continuous on $[a,b]$, then F is differentiable on $[a,b]$ and $F' = f$.

Any function Φ such that $\Phi' = f$ is called a **primitive (antiderivative)** of f.
Two primitives of the same function f differ by a constant.

If F is a primitive of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Integration by parts:
If the functions f and g are continuously differentiable on $[a,b]$, then

$$\int f(x) \cdot g'(x)\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx$$

where $\int f(x)g'(x)dx$ and $\int f'(x)g(x)dx$ represent the set of primitives of fg' and $f'g$, respectively.

Consequence: $\displaystyle\int_a^b f(x)\cdot g'(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)\cdot g(x)\,dx$

Change of variables:

If the function $g:[\alpha,\beta]\to[a,b]$ is a continuously differentiable bijection having the property $g(\alpha)=a$, $g(\beta)=b$ and $f:[a,b]\to\mathbb{R}^1$ is continuous, then

$$\left(\int f(x)\,dx\right)\circ g = \int (f\circ g)(t)\cdot g'(t)\,dt$$

where $\int f(x)dx$ and $\int (f\circ g)(t)\cdot g'(t)\,dt$ represent the set of primitives of f and $(f\circ g)\cdot g'$, respectively.

Consequence: $\displaystyle\int_{g(\alpha)}^{g(\beta)} f(x)\,dx = \int_\alpha^\beta (f\circ g)(t)\cdot g'(t)\,dt$

Improper integrals:

Let f be a function bounded on $[a,\infty)$ and Riemann-Darboux integrable on $[a,b]$ for every $b>a$.

If $\displaystyle\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists, then the **improper integral of first kind** $\displaystyle\int_a^\infty f(x)$ converges.

Let f be a function defined on $(a,b]$ and Riemann-Darboux integrable on $[a+\varepsilon,b]$, for $\varepsilon\in(0,b-a)$.

If $\displaystyle\lim_{\varepsilon\to0^+}\int_{a+\varepsilon}^b f(x)\,dx$ exists, then the **improper integral of second kind** $\displaystyle\int_a^b f(x)\,dx$ converges.

Comparison test for improper integrals of first kind:

Let f and g be defined on $[a,\infty)$ and Riemann-Darboux integrable on $[a,b]$ for every $b>a$.

Suppose that $0\le f(x)\le g(x)$ for all $x\ge a$ and $\displaystyle\int_a^\infty g(x)\,dx$ converges. Then $\displaystyle\int_a^\infty f(x)\,dx$ converges too.

CALCULUS HANDOUT 6 - RIEMANN-DARBOUX INTEGRAL - solved examples

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Ex.1 Find the primitives of the following functions on their maximal domain of definition:

a) $f(x)=x\sqrt{x+1}$ b) $f(x)=\dfrac{3}{1+\sqrt{x-2}}$ c) $f(x)=\dfrac{3x+11}{x^2-x-6}$

Solution: a) Method I: Integration by parts method

$h(x)=x\Rightarrow h'(x)=1;\quad g'(x)=\sqrt{x+1}\Rightarrow g(x)=\frac{2}{3}(x+1)^{\frac{3}{2}}$

$$\Rightarrow \int x\sqrt{x+1}dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{(x+1)^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}}\left(x-\frac{2}{5}(x+1)\right)+C = \frac{2}{3}(x+1)^{\frac{3}{2}}\left(x-\frac{2}{5}x-\frac{2}{5}\right)+C = \frac{2}{3}(x+1)^{\frac{3}{2}}\left(\frac{3}{5}x-\frac{2}{5}\right)+C$$

Method II: Change of variables method

$t=x+1\Leftrightarrow x=t-1\Rightarrow dt=dx$

$$\Rightarrow \int x\sqrt{x+1}dx = \int (t-1)\sqrt{t}dt = \int t\sqrt{t}dt - \int\sqrt{t}dt = \int t^{1+\frac{1}{2}}dt - \int t^{\frac{1}{2}}dt = \frac{t^{2+\frac{1}{2}}}{2+\frac{1}{2}} - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}$$
$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}+C = \frac{2}{3}(x+1)^{\frac{3}{2}}\left(\frac{3}{5}(x+1)-1\right)+C = \frac{2}{3}(x+1)^{\frac{3}{2}}\left(\frac{3}{5}x+\frac{3}{5}-1\right)+C$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}}\left(\frac{3}{5}x-\frac{2}{5}\right)+C$$

b) We consider the following change of variables

$t=\sqrt{x-2}|^2\Leftrightarrow t^2=x-2\Leftrightarrow x=t^2+2\Rightarrow dx=2tdt$

$$\Rightarrow \int\frac{3}{1+\sqrt{x-2}}dx = \int\frac{3}{1+t}\cdot 2tdt = \int\frac{6t}{1+t}dt$$

Consider the change of variables: $u = 1 + t \Leftrightarrow t = u - 1 \Rightarrow du = dt$. Then

$$\int \frac{6t}{1+t} dt = \int \frac{6(u-1)}{u} du = 6 \int du - \frac{1}{u} du = 6u - \ln|u| = 6(1+t) - \ln|1+t| = 6(1+\sqrt{x-2}) - \ln|1+\sqrt{x-2}| + C$$

c) We can show that $\frac{3x+11}{x^2-2-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{4}{x-3} - \frac{1}{x+2}$(check!). Then it results that

$$\int \frac{3x+11}{x^2-x-6} dx = \int \left(\frac{4}{x-3} - \frac{1}{x+2} \right) dx = 4 \int \frac{1}{x-3} dx - \int \frac{1}{x+2} dx = 4\ln|x-3| - \ln|x+2| + C \qquad .$$

Ex.2 We compute the following integrals:

a) $\int_1^2 \ln x dx$ b) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x dx$

Solution: a) Using the integration by parts method, we obtain

$$\int_1^2 \ln x dx = \int_1^2 1 \cdot \ln x dx = \int_1^2 x' \cdot \ln x dx = x \ln x \Big|_1^2 - \int_1^2 x \cdot \frac{1}{x} dx = 2\ln 2 - 1 \ln 1 - \int_1^2 dx$$
$$= 2\ln 2 - 1 \cdot 0 - x\Big|_1^2 = 2\ln 2 - (2-1) = 2\ln 2 - 1.$$

b) We have

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x dx = \int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \cos x dx = \int_0^{\frac{\pi}{2}} \sin^5 x (1 - \sin^2 x) \cos x dx$$

Consider the change of variables $u = \sin x \Rightarrow du = du = \cos x dx$. Then $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^5 x (1 - \sin^2 x) \cos x dx = \int_0^1 u^5 (1-u^2) du = \int_0^1 u^5 du - \int_0^1 u^7 du = \frac{u^6}{6}\Big|_0^1 - \frac{u^8}{8}\Big|_0^1$$
$$= \frac{1^6}{6} - \frac{0^6}{6} - \frac{1^8}{8} + \frac{0^8}{8} = \frac{1}{6} - \frac{1}{8} = \frac{4-3}{24} = \frac{1}{24}$$

Ex.3 We study the convergence of the following improper integrals:

a) $\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$ b) $\int_3^\infty \cos x dx$ c) $\int_0^2 \frac{1}{\sqrt{2-x}} dx$ d) $\int_1^\infty \frac{1}{x+e^x} dx$

Solution:

a) $\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx = \lim_{t \to -\infty} \int_t^0 \frac{1}{\sqrt{3-x}} dx = \lim_{t \to -\infty} (-2\sqrt{3-x})\Big|_t^0 = \lim_{t \to -\infty} (-2\sqrt{3} + 2\sqrt{3-t}) = -2\sqrt{3} + \infty = +\infty$

$\Rightarrow \int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$ is divergent

b) $\int_3^\infty \cos x dx = \lim_{t\to\infty} \int_3^t \cos x dx = \lim_{t\to\infty} \left(\sin x \Big|_3^t \right) = \lim_{t\to\infty} (\sin t - \sin 3) = \lim_{t\to\infty} \sin t - \sin 3$

As $\lim_{t\to\infty} \sin t$ does not exist (check!), it results that $\int_3^\infty \cos x dx$ is divergent.

c) $\int_0^2 \frac{1}{\sqrt{2-x}} dx = \lim_{t \to 2^-} \int_0^t \frac{1}{\sqrt{2-x}} dx = \lim_{t\to 2^-} \left(-\sqrt{2-x}\right)\Big|_0^t = \lim_{t\to 2^-} \left(-\sqrt{2-t} + \sqrt{2}\right) = \sqrt{2}$

We have obtained that $\int_0^2 \frac{1}{\sqrt{2-x}} dx = \sqrt{2}$, therefore $\int_0^2 \frac{1}{\sqrt{2-x}} dx$ is convergent.

d) We study the convergence of the integral $\int\limits_1^\infty \frac{1}{x+e^x}dx$.

$x>1 \Rightarrow \frac{1}{x+e^x} < \frac{1}{e^x} = e^{-x}$

$\int\limits_1^\infty e^{-x}dx = \lim\limits_{t\to\infty}\int\limits_1^t e^{-x}dx = \lim\limits_{t\to\infty}\left(-e^{-x}\right)\Big|_1^t = \lim\limits_{t\to\infty}\left(-e^{-t}+e^{-1}\right) = \frac{1}{e}$

As $\int\limits_1^\infty e^{-x}dx = \frac{1}{e} < \infty$, therefore it is convergent, appplying the comparison test for improper integrals of first kind, we obtain that $\int\limits_1^\infty \frac{1}{x+e^x}dx$ is convergent.

CALCULUS HANDOUT 6 - RIEMANN-DARBOUX INTEGRAL - exercises

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1. Find the primitives of the following functions on their maximal domains of definition:

1. $f(x) = x^2+2x+3$
2. $f(x) = x+\frac{1}{x}$
3. $f(x) = \sin x + \cos x$
4. $f(x) = \frac{1}{\sqrt{1-4x^2}}$
5. $f(x) = \frac{1}{\sqrt{4-x^2}}$
6. $f(x) = \frac{2}{\sin^2 x} + \frac{2}{\cos^2 x}$
7. $f(x) = \frac{1}{\sin^2 x \cdot \cos^2 x}$
8. $f(x) = \frac{1}{x^2+4}$
9. $f(x) = \frac{1}{4x^2+1}$
10. $f(x) = \frac{1}{x^2-1}$
11. $f(x) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$
12. $f(x) = x\sqrt{x} + 2x\sqrt[3]{x^2}$
13. $f(x) = 1+\cos 3x$
14. $f(x) = e^x\cosh(2x)$
15. $f(x) = |x|$
16. $f(x) = \frac{1}{1+\sqrt{x}+\sqrt{x+1}}$
17. $f(x) = \frac{x\,e^{\arctan x}}{\sqrt{(1+x^2)^3}}$
18. $f(x) = \sin(\ln x)$
19. $f(x) = e^{\sqrt{x}}$
20. $f(x) = e^{\arcsin x}$
21. $f(x) = \frac{x^4\arctan x}{1+x^2}$
22. $f(x) = \sqrt{\tan x}$
23. $f(x) = \frac{1-x^2}{(1+x^2)\sqrt{1+x^2}}$
24. $f(x) = \frac{1}{1+x^4}$
25. $f(x) = \frac{x^2}{1+x^4}$
26. $f(x) = \frac{x^2}{1+x^{12}}$
27. $f(x) = \frac{e^x}{e^{2x}+e^{-2x}}$
28. $f(x) = \frac{1}{1+|1-e^x|}$
29. $f(x) = \frac{|x|}{1+|x|}$
30. $f(x) = \frac{1}{3+\cos x}$
31. $f(x) = \frac{1}{1+\sin^2 x}$

2. Compute the following integrals:

1. $\int_0^1 f(x)\,dx$ where $f(x) = \frac{e^{\frac{x-1}{x+1}}}{(1+x)^3}$
2. $\int_0^{\frac{\pi}{2}} f(x)\,dx$ where $f(x) = \ln(\sin x)$
3. $\int_0^{\frac{\pi}{2}} f(x)\,dx$ where $f(x) = \ln(\cos x)$
4. $\int_0^1 f(x)\,dx$ where $f(x) = x\ln(\sin \pi x)$
5. $\int_0^1 f(x)\,dx$ where $f(x) = \frac{x\,e^{\arctan x}}{\sqrt{(1+x^2)^3}}$
6. $\int_2^3 f(x)\,dx$ where $f(x) = \frac{1}{(1+x)\sqrt{1+x+x^2}}$
7. $\int_0^1 f(x)\,dx$ where $f(x) = \frac{x+1}{x^4+x^2+1}$
8. $\int_0^3 f(x)\,dx$ where $f(x) = \sqrt{x+1} + (1-2x)[x-1]$
9. $\int_{-1}^1 f(x)\,dx$ where $f(x) = \begin{cases} \frac{1}{2}e^x & ,x\le 0 \\ \frac{\sqrt{1+x}-1}{x} & ,x>0 \end{cases}$
10. $\int_{-1}^1 f(x)\,dx$ where $f(x) = \begin{cases} -\frac{3}{4}+e^x & ,x\le 0 \\ \frac{\sqrt{x+4}-\sqrt{4}}{x} & ,x>0 \end{cases}$

3. Find recurrence formulae for the following integrals:

1. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$
2. $\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx$
3. $\int_0^{\frac{\pi}{4}} \mathrm{tg}^n x\,dx$
4. $\int_0^1 x^{2n}\sqrt{1-x^2}\,dx$
5. $\int_0^1 x^n \sin \pi x\,dx$

4. Study the convergence of the following improper integrals ($a, b > 0,\ p, q \in \mathbb{R}$):

1. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$
2. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$
3. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$
4. $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$
5. $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,dx$
6. $\displaystyle\int_0^{1} \frac{1}{\sqrt{1-x^2}}\,dx$
7. $\displaystyle\int_0^{1} \frac{1}{x}\,dx$
8. $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}}\,dx$
9. $\displaystyle\int_0^{\infty} \frac{e^{-x}}{1+x^2}\,dx$
10. $\displaystyle\int_0^{1} \frac{e^{-x}}{\sqrt{x}}\,dx$
11. $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$
12. $\displaystyle\int_0^{b} \frac{1}{x^p}\,dx$
13. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$
14. $\displaystyle\int_0^{\infty} \frac{1}{1+|\sin x|}\,dx$
15. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}}\,dx$
16. $\displaystyle\int_0^{\infty} \frac{\arctan x}{x}\,dx$
17. $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$
18. $\displaystyle\int_2^{\infty} \frac{1}{(\ln x)^{\ln x}}\,dx$
19. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\sqrt{x}}\,dx$
20. $\displaystyle\int_0^{\infty} \frac{1}{x^p(1+x^q)}\,dx$
21. $\displaystyle\int_a^{b} \frac{1}{(b-x)^p}\,dx$
22. $\displaystyle\int_2^{\infty} \frac{1}{x^p \ln x}\,dx$
23. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^p}\,dx$
24. $\displaystyle\int_0^{\infty} \frac{x \ln x}{(1+x^2)^p}\,dx$
25. $\displaystyle\int_0^{1} \ln^p \frac{1}{x}\,dx$
26. $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^p}\,dx$