CALCULUS HANDOUT 5 - POWER SERIES. TAYLOR POLYNOMIALS : definitions

POWER SERIES

A series of functions of the form $\sum_{n=0}^{\infty} a_n \cdot x^n$ is called power series.

The Abel-Cauchy-Hadamard theorem: the set of convergence of a power series:

Considering $\omega = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \in [0, +\infty]$ and $R = \begin{cases} \frac{1}{\omega}, & \text{if } \omega \neq 0 \\ +\infty, & \text{if } \omega = 0 \end{cases}$ we have:

- The power series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is absolutely convergent for $|x| < R$. (R called **radius of convergence**)
- The power series diverges $\sum_{n=0}^{\infty} a_n \cdot x^n$ for any x with $|x| > R$.
- For any $r \in (0, R)$ the power series is uniformly convergent on the closed interval $[-r, r]$.

! The ACH theorem does not provide any information about the convergence at $x = \pm R$. Convergence at these points needs to be studied using convergence tests for series of real numbers.

Continuity of the sum of a power series:
The sum of the power series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is a continuous function on $(-R, R)$.

Arithmetics of power series:

Let $\sum_{n=0}^{\infty} a_n \cdot x^n$ and $\sum_{n=0}^{\infty} b_n \cdot x^n$ be power series with radii of convergence R_1 and R_2, where $0 \leq R_1 \leq R_2$.
Then the following power series have radii of convergence larger (or equal) then R_1:

- sum $\sum_{n=0}^{\infty} (a_n + b_n) \cdot x^n$ • scalar product $\sum_{n=0}^{\infty} k \cdot a_n \cdot x^n$ • Cauchy product $\sum_{n=0}^{\infty} c_n \cdot x^n$, $c_n = \sum_{k=0}^{\infty} a_n \cdot b_{n-k}$

More, if $\sum_{n=0}^{\infty} a_n \cdot x^n = f(x)$ and $\sum_{n=0}^{\infty} b_n \cdot x^n = g(x)$, then:

- $\sum_{n=0}^{\infty} (a_n + b_n) \cdot x^n = f(x) + g(x)$ • $\sum_{n=0}^{\infty} (k \cdot a_n) \cdot x^n = k \cdot f(x)$ • $\sum_{n=0}^{\infty} c_n \cdot x^n = f(x) \cdot g(x)$.

Differentiability of the sum of a power series:

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$ and sum $f(x)$.

- f is k-times differentiable and $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot x^{n-k}$ for $|x| < R$.
- For $x = 0$, we obtain $a_k = \frac{f^{(k)}(0)}{k!}$ for $k \in \mathbb{N}$. Hence: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$, for $|x| < R$.
- For small values of x, $f(x) \simeq f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \ldots + \frac{f^{(N)}(0)}{N!}x^N$ for any value of N.

TAYLOR POLYNOMIALS

Let f be an n-times continuously differentiable function on an open interval containing the point a.

The **n-th degree Taylor polynomial of the function f at the point a** is defined by:

$$T_{n,a}f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

? Can we approximate the value $f(x)$ by the value of the Taylor polynomial $T_{n,a}f(x)$?

The first remainder theorem:
Let f be $(n+1)$ times continuously differentiable on an open interval containing the points a and x. Then the difference between f and $T_{n,a}f(x)$ is given by

$$f(x) - T_{n,a}f(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

for some c between a and x.

The error in approximating $f(x)$ by the value of the polynomial $T_{n,a}f(x)$ is the **remainder term**:

$$R_{n,a}f(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

where c lies between a and x. The approximation is good when x is close to a.
The formula $f(x) = T_{n,a}f(x) + R_{n,a}f(x)$ is called **Taylor's formula** (of degree n).
Taylor series representation theorem:
Suppose that the function f has derivatives of all orders on some interval containing the point a and also that $\lim\limits_{n\to\infty} R_{n,a}(x) = 0$ for each x in that interval. Then for any x in that interval, we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a = 0$, the above series are called **MacLaurin** series.

CALCULUS HANDOUT 5 - POWER SERIES. TAYLOR POLYNOMIALS : examples

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Ex.1 We determine the interval of convergence of the following series:
a) $\sum\limits_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}}$; b) $\sum\limits_{n=0}^{\infty} \frac{(x-5)^n}{n^2}$.
Solution: a) Method I: Ratio test.
Denote $a_n = \frac{(-2)^n x^n}{\sqrt{n+3}}$. We compute

$$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}x^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{(-2)^n x^n}\right| = 2|x| \lim_{n\to\infty} \frac{\sqrt{n}\cdot\sqrt{1+\frac{3}{n}}}{\sqrt{n}\cdot\sqrt{1+\frac{4}{n}}} = 2|x|$$

Applying the ratio test, it follows that $\sum\limits_{n=0}^{\infty} a_n$ is convergent for $2|x| < 1$, which is equivalent to $|x| < \frac{1}{2}$, so $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and it is divergent for $|x| > \frac{1}{2}$, so for $x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, +\infty\right)$.
We study the convergence of $\sum\limits_{n=0}^{\infty} a_n$ for $x = \pm\frac{1}{2}$.
If $x = -\frac{1}{2}$, then $\sum\limits_{n=0}^{\infty} a_n = \sum\limits_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ is divergent (apply the integral test or comparison test II).
If $x = \frac{1}{2}$, then $\sum\limits_{n=0}^{\infty} a_n = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ is convergent (apply the Liebniz test for alternating series).
Method II: Apply the Abel-Cauchy-Hadamard theorem
Denote $b_n = \frac{(-2)^n}{\sqrt{n+3}}$. We compute

$$\omega = \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \overline{\lim_{n\to\infty}} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{(-2)^n}\right| = 2 \lim_{n\to\infty} \sqrt{\frac{n\left(1+\frac{3}{n}\right)}{n\left(1+\frac{4}{n}\right)}} = 2$$

As $\omega = 2 \neq 0$, it follows that the radius of convergence of the power series is $R = \frac{1}{\omega} = \frac{1}{2}$.
Therefore $\sum\limits_{n=0}^{\infty} b_n x^n$ is convergent if and only if $|x| < \frac{1}{2}$, and for $x = \pm\frac{1}{2}$ we study the convergence of the given power series separately (see Method I).
Thus, $\sum\limits_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}}$ if convergent for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$.

b) Denote $a_n = \dfrac{(x-5)^n}{n^2}$. We compute

$l = \lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(x-5)^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{(x-5)^n}\right| = |x-5|\lim\limits_{n\to\infty}\dfrac{n^2}{(n+1)^2} = |x-5|$

Applying the ratio test, it follows that $\sum\limits_{n=0}^{\infty} a_n$ is convergent if and only if $l<1$.

$l<1 \Leftrightarrow |x-5|<1 \Leftrightarrow -1<x-5<1 \mid +5 \Leftrightarrow 4<x<6$

If $x=4$, then $\sum\limits_{n=0}^{\infty} a_n = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n^2}$ is convergent (apply the Liebniz test for alternating series).

If $x=6$, then $\sum\limits_{n=0}^{\infty} a_n = \sum\limits_{n=0}^{\infty}\dfrac{1}{n^2}$ is convergent (harmonic series for $p=2>1$).

Therefore, the interval of convergence for the power series is $[4,6]$.

Ex.2 Consider the function $f:(-1,+\infty)\to\mathbb{R}$, $f(x)=\ln(x+1)$.

a) We determine a representation of the function f in power series and its radius of convergence.

b) We find Taylor's formula of order n for the functions f at $a=0$.

c) Using Taylor's formula from b), approximate $\ln(1.1)$ accurate to 3 decimal places.

d) Using Taylor's formula from b), approximate the integral $\int\limits_0^{0.1}\dfrac{\ln(x+1)}{x}dx$ accurate to 4 decimal places.

Solution: a) We have $\dfrac{1}{x+1} = \dfrac{1}{1-(-x)} = 1-x+x^2-x^3+..., \quad |x|<1$.

$(\ln(x+1))' = \dfrac{1}{x+1} \;\Big|\int()dx$

$\Rightarrow \ln(x+1) = \int\dfrac{1}{x+1}dx = \int(1-x+x^2-x^3+...)dx = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+...+C$

As $\ln(1)=C$, it results that $C=0$.

Then $\ln(x+1)=\sum\limits_{n=0}^{\infty}(-1)^{n-1}\cdot\dfrac{x^n}{n}$, and the radius of convergence is $R=1$. (check!)

b) We write Taylor's formula of order n for the function f at $a=0$:

$T_{n,0}f(x) = f(0)+\dfrac{f'(0)}{1!}(x-0)+\dfrac{f''(0)}{2!}(x-0)^2+...+\dfrac{f^{(n)}(0)}{n!}(x-0)^n$

$\Leftrightarrow T_{n,0}f(x) = f(0)+\dfrac{f'(0)}{1!}\cdot x+\dfrac{f''(0)}{2!}\cdot x^2+...+\dfrac{f^{(n)}(0)}{n!}\cdot x^n$

$f(0)=\ln(0+1)=\ln 1=0$

$f'(x)=(\ln(x+1))'=\dfrac{1}{x+1} \Rightarrow f'(0)=\dfrac{1}{0+1}=1$

$f''(x)=\left(\dfrac{1}{x+1}\right)' = -\dfrac{1}{(x+1)^2} \Rightarrow f''(0) = -\dfrac{1}{(0+1)^2} = -1$

$f'''(x)=\left(-\dfrac{1}{(x+1)^2}\right)' = \dfrac{2}{(x+1)^3} \Rightarrow f'''(0)=\dfrac{2}{(0+1)^3}=2$

$f^{(4)}(x)=\left(\dfrac{2}{(x+1)^3}\right)' = -\dfrac{6}{(x+1)^4} \Rightarrow f^{(4)}(0) = -\dfrac{6}{(0+1)^4} = -6$

$f^{(5)}(x)=\left(-\dfrac{6}{(x+1)^4}\right)' = \dfrac{24}{(x+1)^5} \Rightarrow f^{(5)}(0)=\dfrac{24}{(0+1)^5}=24$

$\vdots$

$f^{(n)}(x)=(-1)^{n-1}\cdot\dfrac{(n-1)!}{(x+1)^n} \Rightarrow f^{(n)}(0)=(-1)^{n-1}\cdot(n-1)!$

$\Rightarrow T_{n,0}f(x) = 0+\dfrac{1}{1!}\cdot x-\dfrac{1}{2!}\cdot x^2+\dfrac{2}{3!}\cdot x^3-\dfrac{6}{4!}\cdot x^4+\dfrac{24}{5!}\cdot x^5-...+\dfrac{(-1)^{n-1}(n-1)!}{n!}\cdot x^n$

$\Leftrightarrow T_{n,0}f(x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dfrac{x^5}{5}-...+\dfrac{(-1)^{n-1}\cdot x^n}{n}$

$\Leftrightarrow T_{n,0}f(x) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{x^k}{k}$

The Taylor remainder is $R_{n,0}f(x) = \frac{x^{n+1}}{(n+1)!} \cdot (-1)^n \cdot n! = \frac{(-1)^n \cdot x^{n+1}}{n+1} \xrightarrow[n\to\infty]{} 0$ for $|x| < 1$.

Then, the Taylor series (MacLaurin) associated to the functions is $f(x) = \ln(x+1) = \sum_{n=0}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}$.

(we have obtained the same power series as at a))

c) We can easily see that $\ln(1.1) = \ln(0.1+1) = f(0.1)$.

As the accuracy is 3 decimal places, we have that error $\leq 10^{-3}$.

error $= |f(x) - T_{n,0}f(x)| = |R_{n,0}f(c)| = \left| \frac{x^{n+1}}{(n+1)!} \cdot f^{(n+1)}(c) \right| = \frac{1}{n+1} \left(\frac{x}{c+1} \right)^{n+1}$

error $= |f(0.1) - T_{n,0}f(0.1)| = \frac{0.1^{n+1}}{n+1} \cdot \frac{1}{(c+1)^{n+1}} \leq \frac{10^{-(n+1)}}{n+1}$ (as $c \in (0, 0.1)$)

The smallest $n \in \mathbb{N}$ such that $\frac{10^{-(n+1)}}{n+1} \leq 10^{-3}$ is $n = 2$ (check!). We obtain

$f(0.1) = \ln(1.1) \approx T_{2,0}f(0.1) = \left(x - \frac{x^2}{2} \right) \Big|_{x=0.1} = \frac{1}{10} - \frac{1}{200} = \frac{20-1}{200} = \frac{19}{200} = 0.095$

d) Denote $I = \int_0^{0.1} \frac{\ln(x+1)}{x} dx \approx \int_0^{0.1} \frac{T_{n,0}f(x)}{x} dx$.

As the integral has to be approximated with an accuracy of 4 decimal places, we have that error $\leq 10^{-4}$.

$$\text{error} = \left| I - \int_0^{0.1} \frac{T_{n,0}f(x)}{x} dx \right| = \left| \int_0^{0.1} \frac{\ln(x+1)}{x} dx - \int_0^{0.1} \frac{T_{n,0}f(x)}{x} dx \right| = \left| \int_0^{0.1} \frac{1}{x} \left(\ln(x+1) - T_{n,0}f(x) \right) dx \right|$$

$$= \left| \int_0^{0.1} \frac{R_{n,0}f(c)}{x} dx \right| = \left| \int_0^{0.1} \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} \cdot \frac{1}{(c+1)^{n+1}} dx \right| \leq \int_0^{0.1} \frac{x^n}{n+1} \cdot \frac{1}{(c+1)^{n+1}} dx \leq \int_0^{0.1} \frac{x^n}{n+1} dx =$$

$$= \frac{1}{n+1} \int_0^{0.1} x^n dx = \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} \Big|_0^{0.1} = \frac{10^{-(n+1)}}{(n+1)^2}$$

The smallest $n \in \mathbb{N}$ such that $\frac{10^{-(n+1)}}{(n+1)^2} \leq 10^{-4}$ is $n = 3$ (check!). Then

$$I \approx \int_0^{0.1} \frac{T_{3,0}f(x)}{x} dx = \int_0^{0.1} \frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{x} dx = \int_0^{0.1} dx - \frac{1}{2} \int_0^{0.1} x dx + \frac{1}{3} \int_0^{0.1} x^2 dx = x \Big|_0^{0.1} - \frac{1}{2} \cdot \frac{x^2}{2} \Big|_0^{0.1} + \frac{1}{3} \cdot \frac{x^3}{3} \Big|_0^{0.1} =$$

$$= 0.1 - \frac{1}{4} \cdot (0.1)^2 + \frac{1}{9} \cdot (0.1)^3 = 0.0976$$

Ex.3 We compute $\lim_{x\to 0} \frac{x - \ln(x+1)}{x^2}$ using Taylor series.

Solution: Using example 2a) we have that $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$.

It follows that

$$\lim_{x\to 0} \frac{x - \ln(x+1)}{x^2} = \lim_{x\to 0} \frac{x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right)}{x^2} = \lim_{x\to 0} \frac{x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \ldots \right) = \frac{1}{2}$$

Thus $\lim_{x\to 0} \frac{x - \ln(x+1)}{x^2} = \frac{1}{2}$. (Check, using l'Hospital's rule!)

CALCULUS HANDOUT 5 - POWER SERIES. TAYLOR POLYNOMIALS : exercises

1. Find the interval of convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{(-4)^n x^n}{\sqrt{2n+1}}$
3. $\sum_{n=1}^{\infty} \frac{\ln n x^n}{3^n}$
4. $\sum_{n=1}^{\infty} n^{(-1)^n} x^n$
5. $\sum_{n=0}^{\infty} a^{n^2} x^n,\quad a>0$
6. $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n} (x+1)^n$
7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{1+n^2}} \left(\frac{x}{\sqrt{3}}\right)^n$
8. $\sum_{n=1}^{\infty} n! x^n$
9. $\sum_{n=1}^{\infty} \frac{n!}{2^n} (x-5)^n$
10. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$
11. $\sum_{n=1}^{\infty} \frac{n!}{n^n} (x+3)^n$
12. $\sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{2^n (n+1)^3}$
13. $\sum_{n=0}^{\infty} \frac{x^n}{a^n+b^n},\quad a,b>0$
14. $\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+1}\right)^{n \ln n} x^n$
15. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$
16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n 10^n} (x-2)^n$
17. $\sum_{n=1}^{\infty} \frac{(2n)!}{n!} x^n$
18. $\sum_{n=1}^{\infty} \frac{n!\, x^n}{(a+1)(a+2)\dots(a+n)},\quad a>0$
19. $\sum_{n=2}^{\infty} \frac{\sqrt{n!}\ x^n}{(2+\sqrt{2})(2+\sqrt{3})\dots(2+\sqrt{n})}$
20. $\sum_{n=1}^{\infty} \left[\frac{2^n (n!)^2}{(2n+1)!}\right]^3 \left(\frac{x-1}{2}\right)^n$
21. $\sum_{n=1}^{\infty} \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) x^n$

2. Find Taylor's formula for the following functions. If not stated otherwise, $a=0$ and n is arbitrary.

1. $f(x)=\frac{1}{1-x}$
2. $f(x)=\frac{1}{x}, a=-3$
3. $f(x)=e^x$
4. $f(x)=\cos x$
5. $f(x)=\sin x$
6. $f(x)=\arctan x$
7. $f(x)=\sin 2x$
8. $f(x)=e^{-x}$
9. $f(x)=\sin^2 x$
10. $f(x)=\sin x^2$
11. $f(x)=\sqrt{x+1}$, $n=3$
12. $f(x)=\tan x$, $n=3$
13. $f(x)=\arcsin x$, $n=2$
14. $f(x)=x^2-2x+5$, $n=3$
15. $f(x)=x^{3/2}$, $a=1$, $n=4$
16. $f(x)=\sin x$, $a=\frac{\pi}{2}$, $n=4$
17. $f(x)=\sqrt{x}$, $a=100$, $n=3$
18. $f(x)=(x-4)^{-2}$, $a=5$, $n=5$
19. $f(x)=\tan x$, $a=\frac{\pi}{4}$, $n=4$
20. $f(x)=e^x \sin x$, $n=4$

3. For the first 10 functions given in ex. 2, find the Taylor series representation and its interval of convergence.

4. Use Taylor's formula to approximate the indicated number accurate to three decimal places.

1. $\sqrt[3]{65}$
2. $\sin(0.5)$
3. $\arctan(0.5)$
4. $e^{-0.2}$
5. $\cos(0.3)$

5. Use power series to evaluate the given limits:

1. $\lim_{x\to 0} \frac{1+x-e^x}{x^2}$
2. $\lim_{x\to 0} \frac{1-\cos x}{x(e^x-1)}$
3. $\lim_{x\to 0} \frac{x-\sin x}{x^3 \cos x}$
4. $\lim_{x\to 1} \frac{x^2}{x-1}$
5. $\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\arctan x}$

Extra exercises

6. Use Taylor's formula to approximate the given integrals accurate to four decimal places.

1. $\int_0^1 \frac{\sin x}{x} dx$
2. $\int_0^1 \frac{\arctan x}{x} dx$
3. $\int_0^1 \frac{\cos x}{x} dx$
4. $\int_0^{\frac{1}{2}} \frac{1-e^{-x}}{x}$
5. $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$

7. Find the set of convergence and the sum of the series:

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}$
3. $\sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{x}{2}\right)^n$
4. $\sum_{n=1}^{\infty} n(n-1) x^n$
5. $\sum_{n=1}^{\infty} n\, x^n$
6. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$
7. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)} x^n$
8. $\sum_{n=1}^{\infty} (3n-1) \left(\frac{x}{3}\right)^n$