CALCULUS HANDOUT 4 - SEQUENCES AND SERIES OF FUNCTIONS: definitions

SEQUENCES OF FUNCTIONS

A **sequence of real valued functions** defined on $A \subset \mathbb{R}$ is a function $F : \mathbb{N} \to \{f \mid f : A \to \mathbb{R}\}$.
We write $F(n) = f_n$ and the sequence of functions is denoted by (f_n).
An element $a \in A$ is called **point of convergence** of the sequence (f_n) if the sequence $(f_n(a))$ converges. The set of all points of convergence is called the **set of convergence** of the sequence (f_n).

A function $f : A \to \mathbb{R}$ is called the **limit function** of sequence (f_n) if for any $x \in A$ and $\varepsilon > 0$ there exists $N(x, \varepsilon)$ such that for $n > N(x, \varepsilon)$ we have $|f_n(x) - f(x)| < \varepsilon$. (we write $f_n \xrightarrow[n\to\infty]{} f$ on A.)

The sequence (f_n) is **uniformly convergent** on A to f if for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$ and $x \in A$ we have $|f_n(x) - f(x)| < \varepsilon$. (we write $f_n \xrightarrow[n\to\infty]{u} f$.)

Criteria for uniform convergence:

• **Cauchy's criterion**: The sequence (f_n), defined on A, converges uniformly to a function f defined on A if and only if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that, for any $n, m > N(\varepsilon)$ and any $x \in A$ we have:$|f_n(x) - f_m(x)| < \varepsilon$.

• **2nd criterion**: *Let (f_n) be a sequence of functions defined on A and $f : A \subset \mathbb{R} \to \mathbb{R}$. If there exists a sequence (a_n) of positive real numbers which converges to 0, such that $|f_n(x) - f(x)| \leq a_n$, for any $n \in \mathbb{N}$ and any $x \in A$, then $f_n \xrightarrow[n\to\infty]{u} f$.*

Continuity and uniform convergence:
Let (f_n) be a sequence of functions defined on A which converges uniformly to $f : A \subset \mathbb{R} \to \mathbb{R}$. If all the functions f_n are continuous at a point $a \in A$, then f is continuous at a.

A sequence of functions (f_n) is called **equally continuous** functions A if for any $x \in A$ and $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that if $x' \in A$ and $|x' - x| < \delta$ then $|f_n(x') - f_n(x)| < \varepsilon$ for any $n \in N$.

A sequence of functions (f_n) is called **equally uniformly continuous** on A if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $x', x'' \in A$ and $|x' - x''| < \delta(\varepsilon)$ then $|f_n(x') - f_n(x'')| < \varepsilon$ for any $n \in N$.

A sequence of functions (f_n) is called **equally bounded** on A, if there is $M > 0$, such that $|f_n(x)| < M$ for any $n \in \mathbb{N}$ and $x \in A$.

Arzela-Ascoli Theorem:
Let $I = [a, b]$ be a closed interval and (f_n) an equally continuous and equally bounded sequence of functions defined on I. Then (f_n) contains a subsequence (f_{n_k}) which is uniformly convergent on I.

SERIES OF FUNCTIONS

Let $A \subset \mathbb{R}$ and (f_n) a sequence of functions defined on A.
We say that $\sum_{n=1}^{\infty} f_n$ is a **convergent/divergent series of functions** at the point $a \in A$, if the numerical series $\sum_{n=1}^{\infty} f_n(a)$ is convergent/divergent.

A point $a \in A$ is called **point of convergence** of the series of functions $\sum_{n=1}^{\infty} f_n$ if the series converges at a. The collection of all the points of convergence of the series is called the **set of convergence** of the series $\sum_{n=1}^{\infty} f_n$.
Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on A, and S a function defined on $B \subset A$. The series $\sum_{n=1}^{\infty} f_n$ **converges** to S on B if for any $x \in B$ and any $\varepsilon > 0$ there exists $N = N(x, \varepsilon) > 0$ such that for any $n > N$ we have $|f_1(x) + f_2(x) + \cdots + f_n(x) - S(x)| < \varepsilon$.
If the number N is independent on x, then the series is **uniformly convergent** on B to S.

The series $\sum_{n=1}^{\infty} f_n$ **converges absolutely** on B if the series $\sum_{n=1}^{\infty} |f_n|$ converges on B.
! absolute convergence $\Rightarrow$ convergence

Convergence criteria for series of functions:

The series of functions $\sum_{n=k+1}^{\infty} f_n$ is called the **remainder of order** k of the series $\sum_{n=1}^{\infty} f_n$.

1: The series $\sum_{n=1}^{\infty} f_n$ converges if and only if the remainder of any order k of the series converges.

2: The series $\sum_{n=1}^{\infty} f_n$ converges if and only if the sequence of the sums of remainders tends to 0.

3 (Cauchy): The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for any $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that for $n \geq N$ and $p \geq 1$ we have $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}| < \varepsilon$ for any $x \in A$.

4: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive numbers. If $|f_n(x)| \leq a_n$ for $x \in A$ and $n \in \mathbb{N}$ then the series $\sum_{n=1}^{\infty} f_n$ is uniform convergent.

Ex.1 Consider the sequence of functions $f_n(x) = x^{2n}$ and the functions $f(x) = 0$, $x \in A = \left[0, \frac{1}{3}\right]$ and $g(x) = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases}$. We study if the sequence of functions $f_n(x)$ converges simply and/or uniformly to f on the set A and to g on the set $B = [0,1]$.

Solution: Let $x \in A = \left[0, \frac{1}{3}\right]$.

Then $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{2n} = 0 = f(x)$.

Therefore $f_n \xrightarrow[n\to\infty]{} f$ on A. Moreover, we have that

$|f_n(x) - f(x)| = |x^{2n} - 0| = |x|^{2n} \leq \left(\frac{1}{3}\right)^{2n} = \frac{1}{3^{2n}} \left(\text{as } x \leq \frac{1}{3}\right)$

Denoting $a_n = \frac{1}{3^{2n}}$, we can easily see that $a_n \xrightarrow[n\to\infty]{} 0$. It follows that $f_n \xrightarrow[n\to\infty]{u} f$ on A.

Let $x \in B = [0,1]$. Then, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{2n} = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases} = g(x)$.

It results that $f_n \xrightarrow[n\to\infty]{} g$ pe B. We study the continuity of the function g at 1.

$\lim\limits_{\substack{x\to 1\\ x<1}} f(x) = 0 \neq 1 = f(1)$

It follows that g is not continuous at 1, therefore g is not continuous on $B = [0,1]$.

Thus, $f_n \xrightarrow[n\to\infty]{u\!\!\!/} f$.

Ex.2 For $n \geq 1$, consider the sequence of functions $f_n(x) = \frac{cos^n x}{n^3}$, $x \in \mathbb{R}$ and the series of functions $\sum\limits_{n=1}^{\infty} f_n$. We study if $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent on $\mathbb{R}$.

Soluţie: We have that

$\left|\frac{\cos^n x}{n^3}\right| = \frac{|cos^n x|}{|n^3|} = \frac{|\cos x|^n}{n^3} \leq \frac{1^n}{n^3} = \frac{1}{n^3}$

Denoting $a_n = \frac{1}{n^3}$, we can easily see that $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series, as it is harmonic with $p = 3 > 1$.

It results that $\sum\limits_{n=1}^{\infty} f_n$ is an uniformly convergent series of functions.

Ex.3 Consider the sequence of functions $f_n(x) = \frac{(nx+1)^2}{2n+3}$ and the series $\sum\limits_{n=1}^{\infty} f_n$. We determine the set of convergence for the given series of functions.

Soluţie: Denote with C the set of convergence of the series of functions. If $n \neq 0$, we compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{(nx+1)^2}{2n+3} = \lim_{n\to\infty} \frac{n^2x^2 + 2nx + 1}{2n+3} = \lim_{n\to\infty} \frac{n^2\left(x^2 + \frac{2x}{n} + \frac{1}{n^2}\right)}{n\left(2 + \frac{3}{n}\right)} = +\infty \neq 0$$

If follows that $\sum\limits_{n=1}^{\infty} f_n$ is divergent, for any $x \neq 0$.

For $x = 0$, we have that $\sum\limits_{n=1}^{\infty} f_n = \sum\limits_{n=1}^{\infty} \frac{1}{2n+3} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$, which is divergent, as it is a harmonic series with $p = 1$. (we apply comparison test 2, for example)

In conclusion, $\sum\limits_{n=1}^{\infty} f_n$ is divergent, for any $x \in \mathbb{R}$, which is equivalent to $C = \varnothing$.

CALCULUS HANDOUT 4 - SEQUENCES AND SERIES OF FUNCTIONS: exercises

1. Consider the sequence of functions $f_n(x) = x^n$, defined on $A = [0,1]$, and $f(x) = \begin{cases} 1 & \text{for } x = 1 \\ 0 & \text{for } x \in [0,1) \end{cases}$. Show that the sequence (f_n) is simply convergent to f, but does not converge uniformly to f.

2. Show that the sequence of functions $f_n(x) = \dfrac{\sin n\,x}{n}$, converges uniformly to $f(x) = 0$ on $A = [0, 2\,\pi]$.

3. Consider the sequence of functions defined by $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$ for $n \geq 0$ and the series $\sum\limits_{n=0}^{\infty} f_n(x)$. Show that the set of convergence of this series is $\mathbb{R}$ and the sum of series is $S(x) = \begin{cases} 1+x^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

4. For $n \geq 1$ consider f_n defined on $\mathbb{R}$ as $f_n(x) = \dfrac{\sin^n x}{n^2}$ and the series $\sum\limits_{n=1}^{\infty} f_n$. Show that the series is absolutely convergent on $\mathbb{R}$ and uniformly convergent on $\mathbb{R}$.

5. For $n \geq 1$ consider $f_n(x) = \cos^n x$ and the series $\sum\limits_{n=1}^{\infty} f_n$. Show that the set of convergence is $\mathbb{R} \setminus \{k \cdot \pi\}_{k \in \mathbb{Z}}$. and that the series is absolutely convergent on the set of convergence.

6. For $n \geq 1$ consider the functions $f_n(x) = \dfrac{e^{n \cdot |x|}}{n}$ and the series $\sum\limits_{n=1}^{\infty} f_n$. Show that the set of convergence of the series is empty.

7. Study the convergence of the following series of functions:

1. $\sum\limits_{n=1}^{\infty} \left(\sin \dfrac{1}{n}\right)^x, \quad x \in \mathbb{R}$
2. $\sum\limits_{n=1}^{\infty} \left(1 - \cos \dfrac{1}{n}\right)^x, \quad x \in \mathbb{R}$
3. $\sum\limits_{n=0}^{\infty} \dfrac{x}{x+n+1}, \quad x \in [1,2]$
4. $\sum\limits_{n=1}^{\infty} \dfrac{x}{n} \ln \dfrac{x}{n}, \quad x \in [1,2]$
5. $\sum\limits_{n=1}^{\infty} \dfrac{1}{x^2+n^2}, \quad x \in \mathbb{R}$
6. $\sum\limits_{n=1}^{\infty} \dfrac{n}{x^n}, \quad x \in \mathbb{R}^*$
7. $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x+n}{n^3}, \quad x > 0$
8. $\sum\limits_{n=1}^{\infty} \left[\dfrac{x(x+n)}{n}\right]^n, \quad x \in \mathbb{R}$
9. $\sum\limits_{n=1}^{\infty} \dfrac{x}{(1+x^2)^n}, \quad x \in \mathbb{R}$
10. $\sum\limits_{n=0}^{\infty} \dfrac{x}{(x^2+x^{-2})^n}, \quad x > 0$

Hint: For exercise 7, you can apply one of the tests from series of real numbers (Handout 2).