CALCULUS HANDOUT 3 - Limits. Continuity. Differentiability - definitions, properties

LIMITS

Suppose that $f(x)$ is a function defined on an open interval containing the point a (except possibly not at a itself). We say that L is the **limit** of the function f as x approaches a if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x such that $0 < |x - a| < \delta$ we have $|f(x) - L| < \varepsilon$.

Heine's criterion for the limit:
The function $f : A \subset \mathbb{R} \to \mathbb{R}$ has a limit as x approaches a if and only if for any sequence $(x_n) \subset A \setminus \{a\}$ such that $x_n \to a$ as $n \to \infty$, the sequence $(f(x_n))$ converges.

Cauchy-Bolzano's criterion for the limit:
The function $f : A \subset \mathbb{R} \to \mathbb{R}$ has a limit as x approaches a if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x' - a| < \delta$ and $0 < |x'' - a| < \delta$ implies $|f(x') - f(x'')| < \varepsilon$.

Rules for the limit of a function:
→ If k is a constant, then $\lim\limits_{x\to a} k = k$.
→ If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a}(f(x) \pm g(x)) = L \pm M$.
→ If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a} f(x) \cdot g(x) = L \cdot M$.
→ If $\lim\limits_{x\to a} f(x) = L$, $g(x) \neq 0$ and $\lim\limits_{x\to a} g(x) = M \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

Squeeze rule:
Suppose that the inequality $f(x) \leq g(x) \leq h(x)$ holds for all x in some interval around a, except perhaps at $x = a$. If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} h(x) = L$ then also $\lim\limits_{x\to a} g(x) = L$..

The substitution rule:
Assume that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{y\to L} g(y) = M$. Then $\lim\limits_{x\to a} g(f(x)) = M$.

One-sided limits:
→ L is called the **right-hand limit** of f at a (denoted $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\searrow a} f(x)$), if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \varepsilon$.
→ L is called the **left-hand limit** of f at a (denoted $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\nearrow a} f(x)$) if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \varepsilon$.
! If $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} = L$ then $\lim\limits_{x\to a} f(x) = L$.

Limits at infinity:
→ The number L is the limit of $f(x)$ as x approaches $+\infty$ (denoted by $\lim\limits_{x\to+\infty} f(x) = L$) if for any $\varepsilon > 0$, there exists a number $M > 0$ such that $x > M$ implies $|f(x) - L| < \varepsilon$.
→ The limit $\lim\limits_{x\to-\infty} f(x) = L$ is defined analogously.

Infinite limits:
→ The function f has the right-hand limit $+\infty$ at a (denoted by $\lim\limits_{x\to a^+} f(x) = +\infty$) if for any $M > 0$ there is a $\delta > 0$ such that $f(x) > M$ whenever $a < x < a + \delta$.
→ The function f has the right-hand limit $-\infty$ at a (denoted by $\lim\limits_{x\to a^+} f(x) = -\infty$) if for any $M > 0$ there is a $\delta > 0$ such that $f(x) < -M$ whenever $a < x < a + \delta$.
→ The left-hand limits are defined similarly.

Limit points:
→ The number L is a **limit point** of $f(x)$ at a if there exists a sequence $(x_n) \in A \setminus \{a\}$ such that $\lim\limits_{n\to+\infty} x_n = a$ and $\lim\limits_{n\to+\infty} f(x_n) = L$.
→ The **set of limit points** of $f(x)$ at a is denoted by $\mathcal{L}_a(f)$.
→ $\inf \mathcal{L}_a(f)$ is called the **inferior limit** of f at a and it is denoted by $\varliminf\limits_{x\to a} f(x)$.
→ $\sup \mathcal{L}_a(f)$ is called the **superior limit** of f at a and it is denoted by $\varlimsup\limits_{x\to a} f(x)$.
! The number L is the limit of $f(x)$ as x approaches a if and only if $\varliminf\limits_{x\to a} f(x) = \varlimsup\limits_{x\to a} f(x) = L$.

CONTINUITY

The function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

The function f is continuous at a if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

Heine's criterion for continuity:
The function $f : A \subset \mathbb{R} \to \mathbb{R}$ is continuous at $a \in A$ if and only if for any sequence $(x_n) \subset A$ such that $\lim_{n\to\infty} x_n = a$, the sequence $(f(x_n))$ converges to $f(a)$.

Cauchy-Bolzano's criterion for continuity:
The function $f : A \subset \mathbb{R} \to \mathbb{R}$ is continuous at $a \in A$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x' - a| < \delta$ and $|x'' - a| < \delta$ imply $|f(x') - f(x'')| < \varepsilon$.

Rules for continuity:
→ Sum rule: If f and g are continuous at a then $f + g$ is continuous at a.
→ Product rule: If f and g are continuous at a then $f \cdot g$ is continuous at a.
→ Reciprocal rule: If f is continuous at a and $f(x) \neq 0$, then $\frac{1}{f}$ is continuous at a.
→ Composite rule: If f and g be continuous at a and $f(a)$, respectively, then $g \circ f$ is continuous at a.

The boundedness property:
If f be continuous on the interval $[a, b]$, then f is bounded on $[a, b]$ and attains its bounds.

The intermediate value property:
Let f be continuous on $[a, b]$ and suppose that $f(a) = \alpha$ and $f(b) = \beta$. For every real number γ between α and β there exists a number $c \in (a, b)$ such that $f(c) = \gamma$.

The interval theorem:
Let f be continuous on $I = [a, b]$. Then, $f(I)$ is a closed bounded interval.

The fixed point theorem:
Let $f : [a, b] \to [a, b]$ be a continuous function. Then there is at least one number c such that $f(c) = c$.

The continuity of the inverse function:
Suppose that $f : A \to B$ is a bijection where A and B are intervals. If f is continuous on A, then f^{-1} is continuous on B.

Uniform continuity:
The function $f : A \subset \mathbb{R} \to \mathbb{R}$ is called **uniformly continuous** on A if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x' - x''| < \delta$ implies $|f(x') - f(x'')| < \varepsilon$.

Theorem of uniform continuity:
If $f : A \subset \mathbb{R} \to \mathbb{R}$ is continuous, then f is uniformly continuous on any closed interval $[a, b] \subset A$.

DIFFERENTIABILITY

A function $f : A \subset \mathbb{R} \to \mathbb{R}$ is **differentiable** at $a \in A$ if $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists. The value of this limit is denoted by $f'(a)$ and is called the **derivative** of f at a.

! If f is differentiable at a, then f is continuous at a.

Rules for differentiability:

Sum rule: If $f, g \in \mathcal{D}_a$ then $f+g \in \mathcal{D}_a$ and $(f+g)'(a) = f'(a) + g'(a)$.

Product rule: If $f, g \in \mathcal{D}_a$ then $f \cdot g \in \mathcal{D}_a$ and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

Reciprocal rule: If $f \in \mathcal{D}_a$ and $f(x) \neq 0$, then $\dfrac{1}{f} \in \mathcal{D}_a$ and $\left(\dfrac{1}{f}\right)'(a) = -\dfrac{f'(a)}{f^2(a)}$.

Quotient rule: If $f, g \in \mathcal{D}_a$ and $g(x) \neq 0$, then $\dfrac{f}{g} \in \mathcal{D}_a$ and $\left(\dfrac{f}{g}\right)'(c) = \dfrac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{[g(c)]^2}$.

Composite rule: If $f \in \mathcal{D}_a$ and $g \in \mathcal{D}_{f(a)}$, then $g \circ f \in \mathcal{D}_a$ and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Inverse rule: Suppose that $f : A \to B$ is a continuous bijection where A and B are intervals. If $f \in \mathcal{D}_a$ is differentiable at $a \in A$ and $f'(a) \neq 0$, then f^{-1} is differentiable at $b = f(a)$ and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$.

Local extremes:

A function f has a **local maximum** value at c if c is contained in some open interval I for which $f(x) \leq f(c)$ for each $x \in I$. If $f(x) \geq f(c)$ for each $x \in I$, then f has a **local minimum** value at c.

Fermat's theorem:

If f is differentiable at c and possesses a local maximum or a local minimum at c, then $f'(c) = 0$.

Rolle's theorem:

Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b)$ then there exists $c \in (a, b)$, such that $f'(c) = 0$.

Lagrange's mean value theorem:

Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there exists $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The increasing-decreasing theorem:

If f is differentiable on (a, b) and continuous on $[a, b]$ then

$\to$ $f'(x) > 0$ for all $x \in (a, b)$ implies f is strictly increasing on $[a, b]$;

$\to$ $f'(x) < 0$ for all $x \in (a, b)$ implies f is strictly decreasing on $[a, b]$;

$\to$ $f'(x) = 0$ for all $x \in (a, b)$ implies f is constant on $[a, b]$.

Cauchy's mean value theorem:

Let f and g be differentiable on (a, b) and continuous on $[a, b]$. If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

L'Hôspital's rule:

Suppose that f and g are differentiable on an interval I, $a \in I$ and $g'(x) \neq 0$ (except possibly at a itself). Assume that either **a.** $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or **b.** $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right either exists (as a finite real number) or is $\pm\infty$.

CALCULUS HANDOUT 3 - Limits. Continuity. Differentiability - solved examples

Ex. 1: We study the existence of the limit at the point $a = 0$ for the functions $f : \mathbb{R} \to \mathbb{R}$:

a) $f(x) = \dfrac{|x|}{x}$; b) $f(x) = \cos \dfrac{\pi}{x}$.

Solution:

a) We have $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$. Then

$$l_s = \lim_{\substack{x \to 0 \\ x<0}} f(x) = \lim_{\substack{x \to 0 \\ x<0}} \frac{-x}{x} = -1$$

$$l_d = \lim_{\substack{x \to 0 \\ x>0}} f(x) = \lim_{\substack{x \to 0 \\ x>0}} \frac{x}{x} = 1$$

As $l_s \neq l_d$, it follows that $\lim\limits_{x \to 0} f(x)$ does not exist.

b) Consider $x_n = \dfrac{1}{2n} \xrightarrow[n \to \infty]{} 0$ and $y_n = \dfrac{1}{2n+1} \xrightarrow[n \to \infty]{} 0$. Then

$$f(x_n) = \cos \frac{\pi}{\frac{1}{2n}} = \cos 2n\pi = 1 \xrightarrow[n \to \infty]{} 1$$

$$f(y_n) = \cos \frac{\pi}{\frac{1}{2n+1}} = \cos(2n+1)\pi = \cos(2n\pi + \pi) = \cos \pi = -1 \xrightarrow[n \to \infty]{} -1$$

As $\lim\limits_{n \to \infty} f(x_n) \neq \lim\limits_{n \to \infty} f(y_n)$, applying Heine's theorem for limits, it results that $\lim\limits_{x \to 0} f(x)$ does not exist.

Ex. 2: We study the continuity and differentiability of the functions $f : \mathbb{R} \to \mathbb{R}$.

a) $f(x) = \begin{cases} \dfrac{\sin x}{|x|} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ b) $f(x) = \begin{cases} x^3 \sin \dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Solution: We can easily see that the above functions are both continuous and differentiable on $\mathbb{R} \setminus \{0\}$, as they are both formed out of elementary functions. We study the continuity and differentiability of the functions at $x = 0$.

a) We compute the lateral limits at 0.

$$l_s = \lim_{\substack{x \to 0 \\ x<0}} f(x) = \lim_{\substack{x \to 0 \\ x<0}} \left(-\frac{\sin x}{x}\right) = -\lim_{\substack{x \to 0 \\ x<0}} \frac{\sin x}{x} = -1$$

$$l_d = \lim_{\substack{x \to 0 \\ x>0}} f(x) = \lim_{\substack{x \to 0 \\ x>0}} \frac{\sin x}{x} = 1$$

As $l_s \neq l_d$, it follows that $\lim\limits_{x \to 0} f(x)$ does not exist, and therefore f is not continuous at 0.

Thus, function f is not differentiable at 0.

b) Continuity

$$\left|x^3 \sin \frac{1}{x}\right| = |x^3| \cdot \left|\sin \frac{1}{x}\right| \leq |x^3| \cdot 1 = |x^3|$$

Consider $g : \mathbb{R} \to \mathbb{R}$, $g(x) = |x|^3$. We can easily see that $\lim\limits_{x \to 0} g(x) = 0$. (check!)

Then, it follows that $\lim\limits_{x \to 0} f(x) = 0$. Moreover, $f(0) = 0$.

Therefore, function f is continuous at 0.

Differentiablity

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^3 \sin \frac{1}{x} - 0}{x} = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

$$\left|x^2 \sin \frac{1}{x}\right| = |x^2| \cdot \left|\sin \frac{1}{x}\right| \leq |x^2| \cdot 1 = x^2$$

$$\lim_{x \to 0} x^2 = 0$$

Then, we have that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$, which means that function f is differentiable at 0.

Ex. 3: We prove the inequalities a) $\ln x < x,\ x > 0$ b) $|\cos b - \cos a| \le |b - a|$

Solution:

a) Let $f : (0,\infty) \to \mathbb{R},\ f(x) = x - \ln x$. We compute $f'(x)$. $f'(x) = (x - \ln x)' = x' - (\ln x)' = 1 - \frac{1}{x}$.

We solve the equation: $f'(x) = 0 \Leftrightarrow 1 - \frac{1}{x} = 0 \Leftrightarrow \frac{x-1}{x} = 0 \Leftrightarrow x = 1$.

Computing $f''(x)$, we obtain $f''(x) = (f'(x))' = \left(1 - \frac{1}{x}\right)' = 1' - \left(\frac{1}{x}\right)' = 0 - \left(-\frac{1}{x^2}\right) = \frac{1}{x^2} > 0$ şi $f''(1) = 1 > 0$. Then $x = 1$ is a minimum point, which means that $f(x) \ge f(1)$.

$f(1) = 1 - \ln 1 = 1 - 0 = 1$

We obtain $f(x) \ge 1 > 0 \Leftrightarrow x - \ln x > 0 \Leftrightarrow \ln x < x$, for any $x > 0$.

b) Let $f : [a,b] \to \mathbb{R},\ f(x) = \cos x$. Then $f'(x) = (\cos x)' = -\sin x$.

As f is continuous on $[a,b]$ and differentiable on (a,b), by applying Lagrange's theorem, it follows that there exists $c \in (a,b)$ such that we have $f'(c) = \frac{f(b) - f(a)}{b - a}$. We obtain

$\frac{\cos b - \cos a}{b - a} = -\sin c \Leftrightarrow \left|\frac{\cos b - \cos a}{b - a}\right| = |-\sin c| \Leftrightarrow \left|\frac{\cos b - \cos a}{b - a}\right| = |\sin c| \le 1$

$\Leftrightarrow \frac{|\cos b - \cos a|}{|b - a|} \le 1 \Leftrightarrow |\cos b - \cos a| \le |b - a|$.

Ex. 4: We find the minimum and maximum attained by the function $f(x) = x^3 - 2x^2 - 4x + 2$ on the interval $[-2, 2]$.

Solution: We compute $f'(x)$.

$f'(x) = (x^3 - 2x^2 - 4x + 2)' = (x^3)' - 2\cdot(x^2)' - 4\cdot x' + 2' = 3x^2 - 2\cdot 2x - 4\cdot 1 + 0 = 3x^2 - 4x - 4$

The equation $f'(x) = 0 \Leftrightarrow 3x^2 - 4x - 4 = 0$, has the roots $x_1 = -\frac{2}{3}$ şi $x_2 = 2$. (check!)

As $f(-2) = -6,\ f(2) = -6,\ f\left(-\frac{2}{3}\right) = \frac{94}{27}$, we have:

x	-2			$-\frac{2}{3}$			2
$f'(x)$		+ + +		0		- - -	
$f(x)$	-6		↑	$\frac{94}{27}$	↓		-6

Then $f_{min} = -6$ at $x = -2$ and $x = 2$ and $f_{max} = \frac{94}{27}$ at $x = -\frac{2}{3}$.

CALCULUS HANDOUT 3 - Limits. Continuity. Differentiability - exercises

1. Study the existence of the limits at the point a for the functions:

1. $f(x)=\sin\frac{1}{x},\ \ a=0$
2. $f(x)=\cos\frac{1}{x}-\sin\frac{1}{x},\ \ a=0$
3. $f(x)=x\sin\frac{1}{x},\ \ a=0$
4. $f(x)=x^2\cos^2 x,\ \ a=\infty$
5. $f(x)=x\left[\frac{1}{x}\right],\ \ a=0$
6. $f(x)=\sin^2\frac{1}{x}+\frac{2}{\pi}\operatorname{arctg}\frac{1}{x},\ \ a=0$

2. Study the continuity and differentiability of the following functions (m is a real parameter, $\alpha\in\{0,1,2,3\}$) on their maximal domains of definition:

1. $f(x)=\begin{cases}\sin\frac{1}{x} & ,x\neq 0\\ m & ,x=0\end{cases}$
2. $f(x)=\begin{cases}\cos\frac{1}{x} & ,x\neq 0\\ m & ,x=0\end{cases}$
3. $f(x)=\begin{cases}\frac{1-\cos^3 2x}{x\sin 2x} & ,x\neq 0\\ m & ,x=0\end{cases}$
4. $f(x)=\begin{cases}\frac{1-\cos x}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
5. $f(x)=\begin{cases}\frac{1}{x}\sin\frac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
6. $f(x)=\begin{cases}x^2\cos\frac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
7. $f(x)=\begin{cases}\frac{e^{x^2}-1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
8. $f(x)=\begin{cases}\frac{\ln(1+x^2)}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
9. $f(x)=\begin{cases}x^\alpha\sin\frac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
10. $f(x)=\begin{cases}\sin\frac{1}{x}-\frac{\cos x}{x} & ,x>0\\ m & ,x=0\end{cases}$
11. $f(x)=\begin{cases}x\ln\frac{1-x^2}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$
12. $f(x)=\begin{cases}\frac{1}{x}\operatorname{arctg}\frac{x}{1+x^2} & ,x\neq 0\\ m & ,x=0\end{cases}$

3. Prove the following inequalities:

1. $\frac{b-a}{\cos^2 a}\leq\tan b-\tan a\leq\frac{b-a}{\cos^2 b},\ 0<a<b<\frac{\pi}{2}$
2. $\frac{b-a}{1+b^2}<\arctan\frac{b-a}{1+ab}<\frac{b-a}{1+a^2},\ 0<a<b$
3. $1-\frac{a}{b}<\ln\frac{b}{a}<\frac{b}{a}-1,\ 0<a<b$
4. $\frac{2}{\pi}x<\sin x<x,\ x\in\left(0,\frac{\pi}{2}\right)$
5. $(1+x)^a\geq 1+a\,x,\ x>-1,\ a\in(-\infty,0]\cup[1,\infty)$
6. $(1+x)^a\leq 1+a\,x,\ x>-1,\ a\in(0,1)$
7. $x\ \arctan x>\ln(1+x^2),\ x>0$
8. $(\sin x)^{\sin x}<(\cos x)^{\cos x},\ x\in\left(0,\frac{\pi}{4}\right)$
9. $\sin x+\tan x>2x,\ x\in\left(0,\frac{\pi}{2}\right)$
10. $\arcsin x>x+\frac{x^3}{6},\ x\in(0,1)$
11. $x-\frac{x^3}{3!}<\sin x<x-\frac{x^3}{3!}+\frac{x^5}{5!},\ x\in\left(0,\frac{\pi}{2}\right)$
12. $\sin^2 x+x\ \tan x>2x^2,\ x\in\left(0,\frac{\pi}{2}\right)$

4. Find the maximum and the minimum attained by the following functions on the indicated closed interval.

1. $f(x)=x^2+\frac{16}{x};\ [1,3]$
2. $f(x)=x^2-4x+3;\ [0,2]$
3. $f(x)=\frac{x}{x+1};\ [0,3]$
4. $f(x)=\frac{1-x}{x^2+3};\ [-2,5]$
5. $f(x)=x\sqrt{1-x^2};\ [-1,1]$
6. $f(x)=|x+1|+|x-1|;\ [-2,2]$

Extra exercises

5. Let $g:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function at $a\in I$.
Show that $f=|g|$ is differentiable at a if and only if $g(a)\neq 0$ or $g(a)=g'(a)=0$.
Find the values of the parameter such that the following functions are differentiable on $\mathbb{R}$:
a) $f(x)=|(1+x)^3-3^{at}|$ b) $f(x)=\max\{\lambda+\cos x,\sin x\}$ c) $f(x)=|(x+m)(e^x-1-mt)|$

6. For which values of $a>0$, $a\neq 1$, the following inequality holds:

$$a^x+2^x\geq 3^x+4^x\qquad\forall x\in\mathbb{R}$$

7. What is the equation of a straight line through $(1,0)$ that is tangent to the graph of $f(x)=x+\frac{1}{x}$ at a point in the first quadrant?

8. The period T of oscillation (in seconds) of a simple pendulum of length L (in feet) is given by $T=2\pi\sqrt{L/32}$. What is the rate of change of T with respect to L when $L=4$ feet?

9. What is the shortest possible distance from the parabola $y=x^2$ to the point $(0,1)$?

10. Find two positive real numbers x and y such that their sum is 50 and their product is as large as possible.

11. Find the maximum possible area of a rectangle of perimeter 200 m.

12. Find the dimensions of the rectangle (with sides parallel to the coordinate axes) of maximal area that can be inscribed in an ellipse with equation $\frac{x^2}{25}+\frac{y^2}{9}=1$.