CALCULUS HANDOUT 1 - SEQUENCES: definitions, properties, theorems

A **sequence of real numbers** is a function $n \mapsto a_n$ whose domain is the set of positive integers $\mathbb{N}$ and whose values belong to the set of real numbers $\mathbb{R}$. Usual notation: (a_n).

A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
A sequence (a_n) is **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
A sequence which is either increasing or decreasing is called **monotonic** sequence.

A sequence (a_n) is **bounded** if there exists a number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

The sequence of real numbers (a_n) **converges** to the real number L (or has the **limit** L) if for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for any $n \geq N$.

→ If (a_n) converges to L, then any subsequence (a_{n_k}) of the sequence (a_n) converges to L.
→ Not every sequence has a limit (see for instance, the sequence $a_n = (-1)^n$).
→ If the limit of a sequence (a_n) exists, then it is unique.
→ If the sequence (a_n) converges to L, then it is bounded.

The limit of (a_n) is said to be $+\infty$ if for any $M > 0$ there is N_M such that $a_n > M$ for $n > N_M$.
The limit of (a_n) is said to be $-\infty$ if for any $M > 0$ there is N_M such that $a_n < -M$ for $n > N_M$.

The **set of limit points** (denoted by $\mathcal{L}(a_n)$) of the sequence (a_n) is the collection of points $x \in \mathbb{R}$ for which there exists a subsequence (a_{n_k}) of the sequence (a_n) such that $\lim\limits_{n_k \to \infty} a_{n_k} = x$.
→ The sequence (a_n) converges and $\lim\limits_{n \to +\infty} a_n = L$ if and only if $\mathcal{L}(a_n) = \{L\}$.

The **limit superior** of a sequence (a_n) is $\sup \mathcal{L}(a_n)$. It is usually denoted by $\limsup\limits_{n \to \infty} a_n$ or by $\overline{\lim\limits_{n \to \infty}}\, a_n$.

The **limit inferior** of a sequence (a_n) is $\inf \mathcal{L}(a_n)$. It is usually is denoted by $\liminf\limits_{n \to \infty} a_n$ or $\underline{\lim\limits_{n \to \infty}}\, a_n$.

Bounded monotonic sequence property:
Every bounded monotonic sequence converges (to a finite real number).

Bolzano-Weierstrass Theorem:
Any bounded sequence (a_n) of real numbers contains a convergent subsequence.

Limit laws for sequences:
If the limits $\lim\limits_{n \to \infty} a_n = A$ and $\lim\limits_{n \to \infty} b_n = B$ exist (so A and B are real numbers) then:
1. (scalar product rule) $\lim\limits_{n \to \infty} c a_n = cA$ for any $c \in \mathbb{R}$.
2. (sum rule) $\lim\limits_{n \to \infty} (a_n + b_n) = A + B$
3. (product rule) $\lim\limits_{n \to \infty} a_n b_n = AB$
4. (quotient rule) $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ (assuming that $b_n \neq 0$ and $B \neq 0$)

Squeeze law for sequences:
If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} a_n = L = \lim\limits_{n \to \infty} c_n$ then $\lim\limits_{n \to \infty} b_n = L$ as well.

L'Hospital rule for sequences:
Suppose that $a_n = f(n)$ and $b_n = g(n) \neq 0$ where f and g differentiable functions satisfying either
a. $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) = 0$ or **b.** $\lim\limits_{x \to \infty} f(x) = \pm\infty$ and $\lim\limits_{x \to \infty} g(x) = \pm\infty$. Then
$\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = \lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ (provided that the limit on the right hand side exists as a finite real number or is equal to $\pm\infty$).

Stolz-Cesaro Lemma:
If (a_n) and (b_n) are two sequences such that (b_n) is positive, strictly increasing and unbounded, then $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = \lim\limits_{n \to \infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n}$ (provided that the limit on the right hand side exists).

Cauchy-d'Alembert Lemma:
If (a_n) is a sequence of positive real numbers then $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \sqrt[n]{a_n}$ (provided that one of the two limits exist).

Ex. 1: Let's prove rigorously, based on the definition, that $a_n = \dfrac{1}{\sqrt{n}}$ converges to 0.

Solution:

Let $\varepsilon > 0$.

Then

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon \Big|^2 \Leftrightarrow \frac{1}{n} < \varepsilon^2 \Leftrightarrow n > \frac{1}{\varepsilon^2}.$$

Consider $N(\varepsilon) = \left[\dfrac{1}{\varepsilon^2}\right] + 1$.

Then, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) = \left[\dfrac{1}{\varepsilon^2}\right] + 1 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geq N$, we have $\left|a_n - 0\right| < \varepsilon$.

Thus, $\lim\limits_{n\to\infty} a_n = 0$.

Ex. 2: We compute $\lim\limits_{n\to\infty} \dfrac{3n^2 - 1}{5n^2 + 10n}$.

Solution:

$$\lim_{n\to\infty} \frac{3n^2 - 1}{5n^2 + 10n} = \lim_{n\to\infty} \frac{n^2\left(3 - \frac{1}{n^2}\right)}{n^2\left(5 + \frac{10}{n}\right)} = \lim_{n\to\infty} \frac{3 - \frac{1}{n^2}}{5 + \frac{10}{n}} = \frac{3}{5}$$

Ex. 3: We compute $\underline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} a_n$ for the sequence $a_n = (-1)^n$.

Solution:

$$n = 2k \qquad \Rightarrow a_{2k} = (-1)^{2k} = 1 \underset{n\to\infty}{\longrightarrow} 1$$
$$n = 2k+1 \Rightarrow a_{2k+1} = (-1)^{2k+1} = -1 \underset{n\to\infty}{\longrightarrow} -1$$

The set of limit points is $\mathcal{L}(a_n) = \{-1, 1\}$.

Then

$$\underline{\lim}_{n\to\infty} a_n = \inf \mathcal{L}(a_n) = \inf\{-1, 1\} = -1$$
$$\overline{\lim}_{n\to\infty} a_n = \sup \mathcal{L}(a_n) = \sup\{-1, 1\} = 1$$

Remark: As $\underline{\lim}_{n\to\infty} a_n = -1 \neq 1 = \overline{\lim}_{n\to\infty} a_n$, it follows that $\lim\limits_{n\to\infty} a_n$ doesn't exist.

Ex. 4: We compute $\lim\limits_{n\to\infty} \dfrac{\cos n^2}{2^n}$.

Solution:

Let $n \in \mathbb{N}$. Then

$$-1 \leq \cos n^2 \leq 1 \ \Big| \cdot \frac{1}{2^n}$$
$$\Leftrightarrow -\frac{1}{2^n} \leq \frac{\cos n^2}{2^n} \leq \frac{1}{2^n}$$

As $\lim\limits_{n\to\infty} \left(-\dfrac{1}{2^n}\right) = \lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, applying the squeeze rule, it follows that $\lim\limits_{n\to\infty} \dfrac{\cos n^2}{2^n} = 0$.

Ex. 5: We compute $\lim_{n\to\infty} a_n$, where $a_n = \dfrac{e^{2n}}{n}$.

Solution:

Let $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \dfrac{e^{2x}}{x}$.

We can easily see that $f(n) = a(n) = a_n$.

Then, applying l'Hospital's rule, we obtain

$$\lim_{n\to\infty} \frac{e^{2n}}{n} = \lim_{x\to\infty} \frac{e^{2x}}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1} = +\infty.$$

Ex. 6: We compute $\lim_{n\to\infty} \dfrac{1+\frac{1}{2}+...+\frac{1}{n}}{n}$.

Solution:

Denote $a_n = 1 + \dfrac{1}{2} + ... + \dfrac{1}{n}$ and $b_n = n$.

We have that $b_n \nearrow \infty$ (exercise-prove it).

Then, applying the Stolz-Cesaro lemma, it follows that

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+...+\frac{1}{n}}{n} = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} =$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{2}+...+\frac{1}{n}+\frac{1}{n+1}-\left(1+\frac{1}{2}+...+\frac{1}{n}\right)}{n+1-n} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Ex. 7: We compute $\lim_{n\to\infty} \sqrt[n]{n^2}$.

Solution:

Denote $a_n = n^2$.

Then, for any $n \in \mathbb{N}$, we have $a_n \geq 0$.

Applying Cauchy-d'Alembert's lemma, we obtain

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{n^2} =$$

$$= \lim_{n\to\infty} \left(1+\frac{2}{n}+\frac{1}{n^2}\right) = \lim_{n\to\infty} 1 + 2\lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n^2} = 1 + 2\cdot 0 + 0 = 1.$$

CALCULUS HANDOUT 1 - SEQUENCES: exercises

1. Prove rigorously that the sequence $a_n = \dfrac{1}{n}$ is convergent to 0.

2. Prove rigorously that the sequence $a_n = \dfrac{2n}{5n-3}$ is convergent to $\dfrac{2}{5}$.

3. Prove rigorously that the sequence $a_n = 1 + \left(\dfrac{9}{10}\right)^n$ is convergent to 1.

4. Compute the limits of the following sequences:

   1. $a_n = \left(1 - \dfrac{2}{n^2}\right)^n$
   2. $a_n = \dfrac{\sin n}{3^n}$
   3. $a_n = \dfrac{1+(-1)^n}{\sqrt{n}}$
   4. $a_n = \dfrac{\ln n}{n^x}$, $x \in \mathbb{R}$
   5. $a_n = \dfrac{n^{2005}}{(n+1)^x - n^x}$, $x > 0$
   6. $a_n = \dfrac{1 + \frac{1}{2} + ... + \frac{1}{n+1}}{\ln(n+1)}$
   7. $a_n = \dfrac{1 + \sqrt{2} + ... + \sqrt[n]{n}}{n}$
   8. $a_n = \dfrac{1}{n+1}\left(\dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + ... + \dfrac{1}{\ln(n+2)}\right)$
   9. $a_n = \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{n+1}\dfrac{1}{k}$
   10. $a_n = \dfrac{1^p + 2^p + ... + n^p}{n^{p+1}}$, $p \in \mathbb{N}$
   11. $a_n = \sqrt[n]{n}$
   12. $a_n = \sqrt[n]{n!}$
   13. $a_n = \sqrt[n]{\dfrac{(n!)^2}{(2n+1)!}}$
   14. $a_n = \dfrac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(2n-1)}$

5. Find $\liminf\limits_{n\to\infty} x_n$ and $\limsup\limits_{n\to\infty} x_n$ for each of the following sequences:

   1. $a_n = \begin{cases} 0, & n = 2k+1 \\ 1, & n = 2k \end{cases}$
   2. $a_n = \begin{cases} 1, & n = 3k \\ \dfrac{1}{n}, & n = 3k+1 \\ n, & n = 3k+2 \end{cases}$
   3. $a_n = \cos(n\pi)$
   4. $a_n = \dfrac{n}{n+1}\sin^2\left(\dfrac{n\pi}{4}\right)$
   5. $a_n = \dfrac{[na]}{n+1}$, $a \in \mathbb{R}^\star$
   6. $a_n = \dfrac{(-1)^n}{n} + \dfrac{1+(-1)^n}{2}$
   7. $a_n = \dfrac{n^{(-1)^n}}{n} + \sin^2\dfrac{n\pi}{4}$
   8. $\cos^n \dfrac{2n\pi}{3}$

Extra exercises

6. Let (F_n) be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = F_1 = 1$. Show that $\lim\limits_{n\to\infty}\dfrac{F_{n+1}}{F_n}$ exists and is equal to $\dfrac{1+\sqrt{5}}{2}$.

7. Let the sequence (a_n) be defined recursively as follows:

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 4)$$

Prove by induction that $a_n < 4$ for each n and that (a_n) is an increasing sequence. Find the limit of this sequence.