CALCULUS HANDOUT 11 - LINE INTEGRALS - definitions

ELEMENTARY CURVES

An **elementary curve** is a set of points $C \subset \mathbb{R}^3$ for which there exists a closed interval $[a, b] \subset \mathbb{R}$ and a function $\varphi : [a, b] \to C$ which is bijective on $[a, b)$ and smooth (of class C^1).
The points $A = \varphi(a)$ and $B = \varphi(b)$ are called the **end points** of the curve.
The function φ is called a **parametric representation** of the curve.
The vector $\varphi'(t)$ is tangent to the curve at the point $\varphi(t)$.

An **elementary closed curve** is a curve with parametric representation φ such that $\varphi(a) = \varphi(b)$.

! Any elementary curve possesses an infinity of parametric representations.
! The end points of an elementary curve are independent of the parametric representation of the curve.

The **length** of the elementary curve C with parametric representation $\varphi : [a, b] \to C$ is given by:

$$l = \int_a^b ||\varphi'(t)\| \, dt = \int_a^b \sqrt{\dot{\varphi}_1{}^2(t) + \dot{\varphi}_2{}^2(t) + \dot{\varphi}_3{}^2(t)} \, dt$$

! The curve length is independent of the parametric representation of the curve C.

The **arc length** of the elementary curve C with representation φ is defined as

$$ds = ||\varphi'(t)\| \, dt = \sqrt{\dot{\varphi}_1{}^2(t) + \dot{\varphi}_2{}^2(t) + \dot{\varphi}_3{}^2(t)} \, dt$$

LINE INTEGRALS

Let f be a continuous function defined at least at each point of the curve C, with representation $\varphi : [a, b] \to C$.

Line integral of first type (with respect to the arc length)

$$\int_C f ds = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \cdot \sqrt{\dot{\varphi}_1^2(t) + \dot{\varphi}_2^2(t) + \dot{\varphi}_3^2(t)} \, dt$$

Line integrals of second type (with respect to coordinate variables)

The line integral of f along C with respect to x, y and z:

$$\int_C f(x, y, z) dx = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_1'(t) \, dt$$

$$\int_C f(x, y, z) dy = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_2'(t) \, dt$$

$$\int_C f(x, y, z) dz = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t))\varphi_3'(t) \, dt$$

Green's theorem in the plane

Let R be a closed bounded region in the x, y plane whose boundary C consists of finite many elementary curves. Let $f(x, y)$ and $g(x, y)$ be functions which are continuous and have continuous partial derivatives of first order everywhere in some domain containing R. Then the following equality holds:

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \, dy = \oint_C f \, dx + g \, dy$$

The integration being taken along the entire boundary C of R such that R is on the left as one moves on C.

CALCULUS HANDOUT 11 - LINE INTEGRALS - examples

Ex.1 evaluate the line integrals

$$\text{a)} \int_C f(x,y)\, ds; \qquad \text{b)} \int_C f(x,y)\, dx; \qquad \text{c)} \int_C f(x,y)\, dy,$$

where $f(x,y) = x^2 + y^2$ 'si $C : \; x = 4t - 1, \; y = 3t + 1, \; t \in [-1, 1]$.

Solution:

$$\begin{cases} x = 4t - 1 \\ y = 3t + 1 \end{cases} \Rightarrow \begin{cases} \varphi_1(t) = 4t - 1 \\ \varphi_2(t) = 3t + 1 \end{cases} \Rightarrow \begin{cases} \dot{\varphi_1}(t) = 4 \\ \dot{\varphi_2}(t) = 3 \end{cases}$$

$t \in [-1, 1] \Rightarrow a = -1, b = 1$

a) $\displaystyle\int_C f(x,y)\, ds = \int_a^b f(\varphi_1(t), \varphi_2(t)) \cdot \sqrt{(\dot{\varphi_1}(t))^2 + (\dot{\varphi_2}(t))^2}\, dt$

$$\Rightarrow \int_C f(x,y)\, ds = \int_{-1}^{1} \left((4t-1)^2 + (3t+1)^2\right) \cdot \sqrt{3^3 + 4^2}\, dt = \int_{-1}^{1} \left(16t^2 - 8t + 1 + 9t^2 + 6t + 1\right) \cdot \sqrt{9 + 16}\, dt$$

$$= 5 \int_{-1}^{1} \left(25t^2 - 2t + 2\right)\, dt = 125 \int_{-1}^{1} t^2\, dt - 10 \int_{-1}^{1} t\, dt + 10 \int_{-1}^{1} dt = 125 \cdot \frac{t^3}{3}\Big|_{-1}^{1} - 10 \cdot \frac{t^2}{2}\Big|_{-1}^{1} + 10t\Big|_{-1}^{1}$$

$$= 125 \left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) - 10 \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) + 10(1 - (-1)) = 125 \cdot \frac{2}{3} + 20 = \frac{250 + 60}{3} = \frac{310}{3}$$

b) $\displaystyle\int_C f(x,y)\, dx = \int_a^b f(\varphi_1(t), \varphi_2(t)) \cdot \dot{\varphi_1}(t)\, dt$

$$\Rightarrow \int_C f(x,y)\, dx = \int_{-1}^{1} \left((4t-1)^2 + (3t+1)^2\right) \cdot 4\, dt = 4 \int_{-1}^{1} \left(16t^2 - 8t + 1 + 9t^2 + 6t + 1\right)\, dt$$

$$= 4 \int_{-1}^{1} \left(25t^2 - 2t + 2\right)\, dt = 100 \int_{-1}^{1} t^2\, dt - 8 \int_{-1}^{1} t\, dt + 8 \int_{-1}^{1} dt = 100 \cdot \frac{t^3}{3}\Big|_{-1}^{1} - 8 \cdot \frac{t^2}{2}\Big|_{-1}^{1} + 8t\Big|_{-1}^{1}$$

$$= 100 \left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) - 8 \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) + 8(1 - (-1)) = \frac{200}{3} + 16 = \frac{200 + 48}{3} = \frac{248}{3}$$

c) $\displaystyle\int_C f(x,y)\, dy = \int_a^b f(\varphi_1(t), \varphi_2(t)) \cdot \dot{\varphi_2}(t)\, dt$

$$\Rightarrow \int_C f(x,y)\, dy = \int_{-1}^{1} \left((4t-1)^2 + (3t+1)^2\right) \cdot 3\, dt = 3 \int_{-1}^{1} \left(16t^2 - 8t + 1 + 9t^2 + 6t + 1\right)\, dt$$

$$= 3 \int_{-1}^{1} \left(25t^2 - 2t + 2\right)\, dt = 75 \int_{-1}^{1} t^2\, dt - 6 \int_{-1}^{1} t\, dt + 6 \int_{-1}^{1} dt = 75 \cdot \frac{t^3}{3}\Big|_{-1}^{1} - 6 \cdot \frac{t^2}{2}\Big|_{-1}^{1} + 6t\Big|_{-1}^{1}$$

$$= 75 \left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) - 6 \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) + 6(1 - (-1)) = \frac{150}{3} + 12 = \frac{150 + 24}{3} = \frac{174}{3}$$

Ex.2 Using Green's theorem, evaluate the integral: $\displaystyle\oint_C (x^2 - y^2)\, dx + 2xy\, dy$, where C is the boundary of the region $R = \{(x,y) \in \mathbb{R} \; : \; 0 \le x \le 1 \; , \; 2x^2 \le y \le 2x\}$ (consider positive orientation - counterclockwise).

Solution:

Green's formula: $\displaystyle\oint_C f\, dx + g\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\, dx\, dy$, where

$$\begin{cases} f = x^2 - y^2 \\ g = 2xy \end{cases} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial y} = -2y \\ \dfrac{\partial g}{\partial x} = 2y \end{cases}$$

$$\Rightarrow \oint_C (x^2-y^2)\,dx + 2xy\,dy = \iint_R (2y-(-2y))\,dx\,dy = \int_0^1 \left(\int_{2x^2}^{2x} 4y\,dy\right)\,dx = \int_0^1 \left(4\cdot \frac{y^2}{2}\Big|_{2x^2}^{2x}\right)\,dx$$

$$= \int_0^1 \left(2(2x)^2 - 2(2x^2)^2\right)\,dx = \int_0^1 (8x^2-8x^4)\,dx = 8\int_0^1 x^2\,dx - 8\int_0^1 x^4\,dx$$

$$= 8\cdot \frac{x^3}{3}\Big|_0^1 - 8\cdot\frac{x^5}{5}\Big|_0^1 = \frac{8}{3}-\frac{8}{5} = \frac{40-24}{15} = \frac{16}{15}$$

CALCULUS HANDOUT 11 - LINE INTEGRALS - exercises

1. In the following exercises, evaluate the line integrals:

$$\int_C f(x,y)\,ds \qquad \int_C f(x,y)\,dx \qquad \int_C f(x,y)\,dy$$

   1. $f(x,y)=x+y$ and $C: x=e^t+1,\ y=e^t-1,\ t\in[0,\ln 2]$
   2. $f(x,y)=2x-y$ and $C: x=\sin t,\ y=\cos t,\ t\in[0,\pi/2]$
   3. $f(x,y)=xy$ and $C: x=3t,\ y=t^4,\ t\in[0,1]$
   4. $f(x,y)=xy$ and C is the part of the graph of $y=x^2$ from $A(-1,1)$ to $B(2,4)$
   5. $f(x,y)=y^2$ and C is the part of the graph of $x=y^3$ from $A(-1,-1)$ to $B(1,1)$
   6. $f(x,y)=y\sqrt{x}$ and C is the part of the graph of $y^2=x^3$ from $A(1,1)$ to $B(4,8)$
   7. $f(x,y)=x^2y$ and C consists of the line segments AB and BC where $A(-1,1)$, $B(2,1)$ and $C(2,5)$
   8. $f(x,y)=x^2+y^2$ and C is the arc of the circle $x^2+y^2=1$ from $A(1,0)$ to $B(-1,0)$
   9. $f(x,y)=x+y$ and $C: x=a\cos^3 t,\ y=a\sin^3 t$ between $A(a,0)$ and $B(0,a)$, $a\in\mathbb{R}$
   10. $f(x,y)=\sqrt{y(2-y)}$ and $C: x=t-\sin t,\ y=1-\cos t,\ t\in[0,\pi/2]$

2. Using Green's theorem, evaluate the following integrals (consider positive orientation - counterclockwise):

   1. $\oint_C y\,dx + 2x\,dy$ where C is the boundary of the square $0\le x\le 1,\ 0\le y\le 1$
   2. $\oint_C y^3\,dx + (x^3+3y^2x)\,dy$ where C is the boundary of the region $y=x^2$ and $y=x$, $x\in[0,1]$
   3. $\oint_C 2xy\,dx + (e^x+x^2)\,dy$ where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$
   4. $\oint_C -xy^2\,dx + x^2y\,dy$ where C is the boundary of the region in the first quadrant bounded by $y=1-x^2$
   5. $\oint_C (x+y^2)\,dx + (y+x^2)\,dy$ where C is the square with vertices $(\pm1,\pm1)$
   6. $\oint_C (x^2+y^2)\,dx - 2xy\,dy$ where C is the boundary of the triangle bounded by the lines $x=0$, $y=0$, $x+y=1$
   7. $\oint_C (y+e^x)\,dx + (2x^2+\cos y)\,dy$ where C is the boundary of the triangle with vertices $(0,0)$, $(1,1)$, $(2,0)$
   8. $\oint_C (-y^2+e^{e^x})\,dx + \arctan y\,dy$ where C is the boundary of the region between the parabolas $y=x^2$, $x=y^2$
   9. $\oint_C y^2\,dx + (2x-3y)\,dy$ where C is the circle $x^2+y^2=9$
   10. $\oint_C (x-y)\,dx + y\,dy$ where C is the boundary of the region between Ox and the graph of $y=\sin x$, $x\in[0,\pi]$

Extra exercises

3. Compute the line integrals of first type $\int_C f(x,y,z)\,ds$ for:

   1. $f(x,y,z) = xy^3$ and C is the line segment $y = 2x$ in the plane Oxy from $A(-1,-2,0)$ to $B(1,2,0)$
   2. $f(x,y,z) = (x^2+y^2+z^2)^2$ and C is the helix $\varphi(t) = (\cos t, \sin t, t)$ from $A(1,0,0)$ to $B(1,0,6\pi)$
   3. $f(x,y,z) = xyz$ and C is the straight line segment from $A(1,-1,2)$ to $B(3,2,5)$
   4. $f(x,y,z) = 2x + 9xy$ and $C:\ x = t,\ y = t^2,\ z = t^3,\ t \in [0,1]$
   5. $f(x,y,z) = xy$ and C is the elliptical helix $x = 4\cos t,\ y = 9\sin t,\ z = 7t,\ t \in [0, 5\pi/2]$

4. Compute the line integral of second type $\int_C x^2y\,dx + (x-z)\,dy + xyz\,dz$ in each of the following cases:

   1. C is the arc of parabola $y = x^2$ in the plane $z = 2$ from $A(0,0,2)$ to $B(1,1,2)$
   2. C is the straight line segment $x = y,\ z = 2$, from $A(0,0,2)$ to $B(1,1,2)$