CALCULUS HANDOUT 11 - DOUBLE AND TRIPLE INTEGRALS - definitions

JORDAN MEASURABLE SETS IN $\mathbb{R}^2$

Consider the set of one dimensional bounded intervals of the form (a, b), $[a, b)$, $(a, b]$, $[a, b]$, where $a, b \in \mathbb{R}$.
The cartesian product $\Delta = I_1 \times I_2$ of two intervals of this type will be called **rectangle** in $\mathbb{R}^2$.
The **area** of such a rectangle Δ is defined by $area(\Delta) = length(I_1) \cdot length(I_2)$.

Consider the set $\mathcal{P}$ of all finite reunions of rectangles Δ: $P \in \mathcal{P}$ iff there exist $\Delta_1, \Delta_2, ..., \Delta_n$ such that $P = \bigcup_{i=1}^{n} \Delta_i$.

- If $P_1, P_2 \in \mathcal{P}$, then $P_1 \cup P_2 \in \mathcal{P}$ and $P_1 \setminus P_2 \in \mathcal{P}$.
- For any $P \in \mathcal{P}$ there exist $\Delta_1, \Delta_2, ..., \Delta_n$ such that $P = \bigcup_{i=1}^{n} \Delta_i$ and $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$.

The **area** of a set $P \in \mathcal{P}$ is defined by $area(P) = \sum_{i=1}^{n} area(\Delta_i)$ where $P = \bigcup_{i=1}^{n} \Delta_i$ and $\Delta_1, \Delta_2, ..., \Delta_n$ are disjoint.

- The area defined in this way for $P \in \mathcal{P}$ satisfies:
  - $area(P) > 0$ for $P \in \mathcal{P}$
  - if $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 = \emptyset$, then $area(P_1 \cup P_2) = area(P_1) + area(P_2)$
  - it is independent on the decomposition of P in finite union of disjoint intervals.

For a bounded set $A \subset \mathbb{R}^2$, we define $area_i(A) = \sup_{P \subset A, P \in \mathcal{P}} area(P)$ and $area_e(A) = \inf_{P \supset A, P \in \mathcal{P}} area(P)$.
A bounded set $A \subset \mathbb{R}^2$ is said **Jordan measurable** if $area_i(A) = area_e(A)$.
The **area** of a Jordan measurable set $A \subset \mathbb{R}^2$ is defined as $area(A) = area_i(A) = area_e(A)$.

- If A_1 and A_2 are Jordan measurable sets, then $A_1 \cup A_2$ and $A_1 \setminus A_2$ are Jordan measurable.
- If $A_1 \cap A_2 = \emptyset$, then $area(A_1 \cup A_2) = area(A_1) + area(A_2)$.

THE RIEMANN-DARBOUX INTEGRAL OF FUNCTIONS OF TWO VARIABLES

Let A be a given bounded and Jordan measurable subset of $\mathbb{R}^2$.

A **partition** P of A is a finite set of disjoint Jordan measurable subsets A_i, $i = \overline{1, n}$ of A satisfying: $\bigcup_{i=1}^{n} A_i = A$.

The **diameter** of the set A_i is the number $d(A_i)$ defined by $d(A_i) = \max_{(x',y'),(x'',y'') \in A_i} \sqrt{(x' - x'')^2 + (y' - y'')^2}$.
The **norm of the partition** P is the number $\nu(P) = \max\{d(A_1), d(A_2), \cdots, d(A_n)\}$.
Let $f : A \to \mathbb{R}^1$ be a bounded function.
Then f is bounded on each part A_i and has a least upper bound M_i and a greatest lower bound m_i on A_i.

The **upper Darboux sum** of f related to P is $U_f(P) = \sum_{i=1}^{n} M_i \cdot area(A_i)$, where $M_i = \sup\{f(x, y) \,|\, (x, y) \in A_i\}$.

The **lower Darboux sum** of f related to P is $L_f(P) = \sum_{i=1}^{n} m_i \cdot area(A_i)$, where $m_i = \inf\{f(x, y) \,|\, (x, y) \in A_i\}$.

The **Riemann sum** of f related to P is defined by $\sigma_f(P) = \sum_{i=1}^{n} f(\xi_i, \eta_i) \cdot area(A_i)$ where $(\xi_i, \eta_i) \in A_i$.

- The following inequalities hold $L_f(P) \leq \sigma_f(P) \leq U_f(P)$.

As f is bounded above and below on A, there exist numbers m and M with $m \leq f(x, y) \leq M$ for all $(x, y) \in A$.
For any partition P of A we have

$$m \cdot area(A) = m \cdot \sum_{i=1}^{n} area(A_i) \leq L_f(P) \leq U_f(P) \leq M \cdot \sum_{i=1}^{n} area(A_i) = M \cdot area(A)$$

Hence, the sets $L_f = \{L_f(P) \,|\, P \text{ is a partition of } A\}$ and $U_f = \{U_f(P) \,|\, P \text{ is a partition of } A\}$ are bounded.
We can therefore consider $\mathcal{L}_f = \sup_P L_f$ and $\mathcal{U}_f = \inf_P U_f$.

- If f is defined and bounded on A, then $\mathcal{L}_f \leq \mathcal{U}_f$.

A function f defined and bounded on A is **Riemann-Darboux integrable** on A if $\mathcal{L}_f = \mathcal{U}_f$.
This common value is denoted by

$$\iint_A f(x, y)\, dx\, dy$$

and it is called the **double integral** of f.

Classes of Riemann-Darboux integrable functions:

• If f is continuous on $\overline{A}$ and $\overline{A}$ is Jordan measurable, then f is Riemann-Draboux integrable on $\overline{A}$.

A function f is called **piecewise-continuous** on A if there exists a partition $P = \{A_1, \cdots, A_n\}$ of A and continuous functions f_i, $i = \overline{1, n}$ defined on A_i such that $f(x) = f_i(x)$ for $x \in Int(A_i)$.

• A piecewise-continuous function is Riemann-Darboux integrable and $\iint\limits_A f(x,y)\,dx\,dy = \sum\limits_{i=1}^{n} \iint\limits_{A_i} f_i(x,y)\,dx\,dy$.

Properties of the Riemann-Darboux integral:

If f and g are Riemann-Darboux integrable on A, then all the integrals below exist and the following hold:

(1) $\iint\limits_A [\alpha f(x,y) + \beta g(x,y)]\,dx\,dy = \alpha \iint\limits_A f(x,y)\,dx\,dy + \beta \iint\limits_A g(x,y)\,dx\,dy,\ \alpha, \beta \in \mathbb{R}^1$

(2) $\iint\limits_A f(x,y)\,dx\,dy = \iint\limits_{A_1} f(x,y)\,dx\,dy + \iint\limits_{A_2} f(x,y)\,dx\,dy$ where $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$

(3) if $f(x,y) \le g(x,y)$ on A, then $\iint\limits_A f(x,y)\,dx\,dy \le \iint\limits_A g(x,y)\,dx\,dy$

(4) $\left| \iint\limits_A f(x,y)\,dx\,dy \right| \le \iint\limits_A |f(x,y)|\,dx\,dy$

The mean value theorem:

Let $f : A \to \mathbb{R}^1$ be integrable on A and satisfying $m \le f(x,y) \le M$ for any $(x,y) \in A$.

Then $m \cdot area(A) \le \iint\limits_A f(x,y)\,dx\,dy \le M \cdot area(A)$.

Riemann-Darboux integral calculus when A is rectangular:

Assume that A is a rectangle, $A = [a,b] \times [c,d]$ and $f : A \to \mathbb{R}^1$ is a continuous function. Then:

$$\iint\limits_A f(x,y)\,dx\,dy = \int\limits_a^b \left(\int\limits_c^d f(x,y)\,dy \right) dx = \int\limits_c^d \left(\int\limits_a^b f(x,y)\,dx \right) dy$$

Therefore, the computation of a double integral on a rectangular domain reduces to the computation of two successive (or *iterated*) single-variable integrals.

Riemann-Darboux integral calculus when A is not a rectangle:

Let A the set defined by

$$A = \{((x,y) \,|\, x \in [a,b] \ \text{ and } \ y \in [g(x), h(x)]\}$$

where g, h are continuous functions satisfying $g(x) \le h(x)$ for every $x \in [a,b]$.

For a continuous function $f : A \to \mathbb{R}^1$ we have:

$$\iint\limits_A f(x,y)\,dx\,dy = \int\limits_a^b dx \int\limits_{g(x)}^{h(x)} f(x,y)\,dy$$

Change of variables in double integrals:

If $A, B \subset \mathbb{R}^2$ are Jordan measurable sets, $T : B \to A$ is a bijection such that T and T^{-1} have continuous partial derivatives and $f : A \to \mathbb{R}^1$ is an integrable function, then the following equality holds:

$$\iint\limits_A f(x,y)\,dx\,dy = \iint\limits_B f(x(\xi,\eta), y(\xi,\eta)) \left\| \begin{matrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{matrix} \right\| d\xi\,d\eta$$

CALCULUS HANDOUT 11 - DOUBLE AND TRIPLE INTEGRALS - examples

Ex.1 Compute the double integral $\iint\limits_D (6xy^2)\,dx\,dy$ on the rectangle $D = [2,4] \times [1,2]$.

Solution:

Method I:

$$\iint\limits_D (6xy^2)\,dx\,dy = \int_1^2 \left(\int_2^4 6xy^2\,dx \right) dy = \int_1^2 6y^2 \left(\int_2^4 x\,dx \right) dy$$
$$= \int_1^2 6y^2 \left(\frac{x^2}{2}\Big|_2^4 \right) dy = \int_1^2 3y^2 \left(4^2 - 2^2\right) dy$$
$$= \int_1^2 3 \cdot 12y^2\,dy = 36 \cdot \int_1^2 y^2\,dy$$
$$= 36 \cdot \frac{y^3}{3}\Big|_1^2 = 12 \cdot (2^3 - 1^3)$$
$$= 12 \cdot 7 = 84$$

Methd II:

$$\iint\limits_D (6xy^2)\,dx\,dy = \int_2^4 \left(\int_1^2 6xy^2\,dy \right) dx = \int_2^4 \left(\int_1^2 6xy^2\,dy \right) dx$$
$$= \int_2^4 6x \left(\int_1^2 y^2\,dy \right) dx = \int_2^4 6x \left(\frac{y^3}{3}\Big|_1^2 \right) dx$$
$$= \int_2^4 2x(2^3 - 1^3)\,dx = \int_2^4 2 \cdot 7x\,dx$$
$$= 14 \cdot \int_2^4 x dx = 14 \cdot \frac{x^2}{2}\Big|_2^4$$
$$= 7 \cdot (4^2 - 2^2) = 7 \cdot 12 = 84$$

Ex.2 Compute the double integral $\iint\limits_D (4xy - y^3)\,dx\,dy$, where D is the domain bounded by the curves $y = \sqrt{x}$ and $y = x^3$.

Solution:

Determine the domain D.

$\sqrt{x} = x^3 \ |^2 \Leftrightarrow x = x^6 \Leftrightarrow x(x^5 - 1) = 0 \Leftrightarrow x = 0$ or $x = 1$

$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \text{ and} x^3 \le y \le \sqrt{x}\}$

Then, we obtain that

$$\iint_D (4xy - y^3)\,dx\,dy = \iint_D 4xy\,dx\,dy - \iint_D y^3\,dx\,dy = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx - \int_0^1 \int_{x^3}^{\sqrt{x}} y^3\,dy\,dx$$

$$= \int_0^1 4x \left(\int_{x^3}^{\sqrt{x}} y\,dy \right) dx - \int_0^1 \left(\int_{x^3}^{\sqrt{x}} y^3\,dy \right) dx = \int_0^1 4x \cdot \left(\frac{y^2}{2}\Big|_{x^3}^{\sqrt{x}} \right) dx - \int_0^1 \left(\frac{y^4}{4}\Big|_{x^3}^{\sqrt{x}} \right) dx$$

$$= \int_0^1 2x \cdot ((\sqrt{x})^2 - (x^3)^2)\,dx - \int_0^1 \left(\frac{(\sqrt{x})^4}{4} - \frac{(x^3)^4}{4} \right) dx = 2\int_0^1 x(x - x^6)\,dx - \frac{1}{4}\int_0^1 (x^2 - x^{12})\,dx$$

$$= 2\int_0^1 x^2\,dx - 2\int_0^1 x^7\,dx - \frac{1}{4}\int_0^1 x^2\,dx + \frac{1}{4}\int_0^1 x^{12}\,dx = \frac{7}{4} \cdot \frac{x^3}{3}\Big|_0^1 - 2 \cdot \frac{x^8}{8}\Big|_0^1 + \frac{1}{4} \cdot \frac{x^{13}}{13}\Big|_0^1$$

$$= \frac{7}{4} \cdot \frac{1}{3} - \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{13} = \frac{91 - 39 + 3}{156} = \frac{55}{156}$$

Ex.3 Compute the double integral $\iint_D e^{x^2+y^2}\,dx\,dy$, where D is the unit disk $x^2 + y^2 \leq 1$.

Solution:

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{x^2+y^2}$.

We use the polar coordinates

$$\begin{cases} x = r\cos\theta & 0 \leq r \leq 1 \\ y = r\sin\theta & 0 \leq \theta \leq 2\pi \end{cases} \Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow I = \iint_D f(x, y)\,dx\,dy = \iint_{D_1} f(x(r, \theta), y(r\theta)) \cdot |J(r, \theta)|\,dx\,dy$$

$D_1 = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi\}$

$$J(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

$$\Rightarrow I = \int_0^{2\pi} \left(\int_0^1 r \cdot e^{r^2}\,dr \right) d\theta = \int_0^{2\pi} \left(\frac{1}{2} e^{r^2} \right) d\theta = \frac{1}{2} \int_0^{2\pi} (e^1 - e^0)\,d\theta$$

$$= \frac{1}{2}(e - 1) \cdot \theta\Big|_0^{2\pi} = \frac{1}{2}(e - 1)(2\pi - 0) = \pi(e - 1)$$

Ex.4 Compute the triple integral $\iiint_D 8xyz\,dx\,dy\,dz$, where $D = [2, 3] \times [1, 2] \times [0, 1]$.

Solution:

$$\iiint_D 8xyz\,dx\,dy\,dz = \int_0^1 \int_1^2 \int_2^3 8xyz\,dx\,dy\,dz = \int_0^1 \int_1^2 8yz \left(\int_2^3 x\,dx \right) dy\,dz = \int_0^1 \int_1^2 8yz \cdot \left(\frac{x^2}{2}\Big|_2^3 \right) dy\,dz$$

$$= \int_0^1 \int_1^2 4yz(3^2 - 2^2)\,dy\,dz = \int_0^1 \int_1^2 20yz\,dy\,dz = \int_0^1 20z \left(\int_1^2 y\,dy \right) dz$$

$$= \int_0^1 20z \cdot \left(\frac{y^2}{2}\Big|_1^2 \right) dz = \int_0^1 10z(2^2 - 1^2)\,dz$$

$$= \int_0^1 30z\,dz = 30 \int_0^1 z\,dz = 30 \cdot \frac{z^2}{2}\Big|_0^1$$

$$= 15 \cdot (1^2 - 0^2) = 15$$

CALCULUS HANDOUT 11 - DOUBLE AND TRIPLE INTEGRALS - exercises

1. Compute the following double integrals on the given rectangles:
   1. $\iint_\Delta (3x + 4y)\, dx\, dy$, if $\Delta = [0, 2] \times [0, 4]$
   2. $\iint_\Delta xy\, dx\, dy$, if $\Delta = [1, 2] \times [1, 2]$
   3. $\iint_\Delta x^2 y\, dx\, dy$, if $\Delta = [0, 3] \times [0, 2]$
   4. $\iint_\Delta (xy + 7x + y)\, dx\, dy$, if $\Delta = [0, 3] \times [0, 3]$
   5. $\iint_\Delta (x^3 y - xy^3)\, dx\, dy$, if $\Delta = [1, 3] \times [-3, -1]$
   6. $\iint_\Delta \ln(x + y)\, dx\, dy$, if $\Delta = [0, 1] \times [1, 2]$
   7. $\iint_\Delta \dfrac{\cos y}{1 + \sin x \cdot \sin y}\, dx\, dy$, if $\Delta = \left[0, \dfrac{\pi}{2}\right] \times [0, \pi]$
   8. $\iint_\Delta \dfrac{1}{(1 + xy)^2}\, dx\, dy$, if $\Delta = [0, 1] \times [0, 1]$
   9. $\iint_\Delta \dfrac{y}{1 + xy}\, dx\, dy$, if $\Delta = [0, 1] \times [0, 1]$
   10. $\iint_\Delta \dfrac{\sin^2 x}{\cos^2 y}\, dx\, dy$, if $\Delta = \left[0, \dfrac{\pi}{2}\right] \times \left[0, \dfrac{\pi}{4}\right]$

2. Compute the following double integrals:
   1. $\iint_D xy\, dx\, dy$, if D is bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.
   2. $\iint_D x^2\, dx\, dy$, if D is bounded by the parabola $y = 2 - x^2$ and the line $y = -4$.
   3. $\iint_D x\, dx\, dy$, if D is bounded by the parabolas $y = x^2$ and $y = 8 - x^2$.
   4. $\iint_D x\, dx\, dy$, if D is bounded by the x-axis and the curve $y = \sin x$, $0 \le x \le \pi$.
   5. $\iint_D \sin x\, dx\, dy$, if D is bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$.
   6. $\iint_D xy\, dx\, dy$, if D is the first quadrant quarter of the circle bounded by $x^2 + y^2 = 1$ and the axes.
   7. $\iint_D \dfrac{1}{y}\, dx\, dy$, if D is the triangle bounded by the lines $y = 1$, $x = e$ and $y = x$.
   8. $\iint_D \dfrac{x^2}{\sqrt{x^2 + y^2}}\, dx\, dy$, if D is bounded by the lines $x = 0$, $y = 1$, $y = \sqrt[3]{2}$ and $y = x$.
   9. $\iint_D (1 - y)\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + (y - 1)^2 \le 1,\ y \le x^2,\ x \ge 0\}$.
   10. $\iint_D \arcsin \sqrt{x + y}\, dx\, dy$, if D is bounded by the lines $x + y = 0$, $x + y = 1$, $y = -1$ and $y = 1$.
   11. $\iint_D \dfrac{1}{xy}\, dx\, dy$, if D is bounded by the curves $y^2 = 2x$, $2 < x < a$ and $xy = 4$, $y > 0$.
   12. $\iint_D \sqrt{xy}\, dx\, dy$, if D is bounded by the curves $y = x^3$, $y = x^2$, $x > 0$.
   13. $\iint_D \sqrt{x + y}\, dx\, dy$, if D is the interior of the triangle of vertices 0, $A(1, 2)$ and $B(3, 2)$.
   14. $\iint_D \cos(x + y)\, dx\, dy$, if D is bounded by $x = 0$, $y = \pi$, $y = x$.
   15. the area of the domain D bounded by $y = x^3 + 1$ and $y = 3x^2$.
   16. the area of the domain D bounded by $y = x^2 - 1$ and $y = \dfrac{1}{x^2 + 1}$.
   17. the area of the domain D bounded by $y = x^2 - 2x$ and $y = \sin x$.

3. Using polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), compute the following integrals:

   1. the area of $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le R^2\}$.
   2. $\iint\limits_D \frac{1}{1 + x^2 + y^2}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid y \in [0, 1],\ 0 \le x \le \sqrt{1 - y^2}\}$.
   3. $\iint\limits_D \frac{1}{\sqrt{4 - x^2 - y^2}}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1],\ 0 \le y \le \sqrt{1 - x^2}\}$.
   4. $\iint\limits_D (x^2 + y^2)^{3/2}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 2],\ 0 \le y \le \sqrt{4 - x^2}\}$.
   5. $\iint\limits_D (x^2 + y^2)\, dx\, dy$, where D is bounded by $x^2 + y^2 = 1$, $y = x\sqrt{3}$, $x = y\sqrt{3}$, and $x > 0$.
   6. $\iint\limits_D \frac{y^2}{x^2}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 2x\}$.
   7. $\iint\limits_D \frac{x + y}{x^2 + y^2}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid x \le x^2 + y^2 \le 1,\ 0 \le y \le x\}$.
   8. $\iint\limits_D (x^2 + y^2)\, dx\, dy$, where D is bounded by the curves $x^2 + y^2 = x$ and $x^2 + y^2 = 2x$.
   9. $\iint\limits_D e^{x^2+y^2}\, dx\, dy$, where D is given by $x^2 + y^2 \le a^2$.
   10. $\iint\limits_D \sqrt{\frac{1 - x^2 - y^2}{1 + x^2 + y^2}}\, dx\, dy$, where D is given by $x^2 + y^2 \le 1$, $x \ge 0$, $y \ge 0$.

4. Compute the triple integral $\iiint\limits_D f(x, y, z)\, dx\, dy\, dz$ for the following functions:

   1. $f(x, y, z) = xy\sin z$, where $D = [0, \pi] \times [0, \pi] \times [0, \pi]$.
   2. $f(x, y, z) = xz + y$, where $D = [-1, 1] \times [0, 2] \times [1, 3]$.
   3. $f(x, y, z) = xy$, where D is given by $1 \le x \le 2$, $-2 \le y \le -1$, $0 \le z \le \frac{1}{2}$.
   4. $f(x, y, z) = \frac{1}{(1 + x + y + z)^3}$, where D is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
   5. $f(x, y, z) = xy\sqrt{z}$, where D is bounded by $z = 0$, $z = y$, $y = x^2$, $y = 1$.
   6. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, where D is given by $x^2 + y^2 + z^2 \le 2$.
   7. $f(x, y, z) = 1$, where $D = \{(x, y, z) \in \mathbb{R}^3 \mid (x - a)^2 + (y - b)^2 + (z - c)^2 \le R^2\}$.
   8. $f(x, y, z) = y^2$, where $D = \{(x, y, z) \in \mathbb{R}^3 \mid y \ge 0 \quad \text{and} \quad x^2 + y^2 + z^2 \le 1\}$.
   9. $f(x, y, z) = \frac{xyz}{x^2 + y^2 + z^2}$, where $D \subset \mathbb{R}^3$ is bounded by $(x^2 + y^2 + z^2)^2 = xy$, where $z \ge 0$.
   10. $f(x, y, z) = x^2 + y^2 + z^2$, where $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le z \in [0, 1]\}$.
   11. $f(x, y, z) = z$, where $D = \left\{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$.
   12. $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, where D is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.
   13. $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + (z - 2)^2}}$, where D is bounded by $x^2 + y^2 \le 1$ and $-1 \le z \le 1$.
   14. $f(x, y, z) = z$, where D is given by $x^2 + y^2 \le z^2$, $0 \le z \le 1$.