CALCULUS HANDOUT 10
HIGHER ORDER DIFFERENTIABILITY. LOCAL EXTREMA - definitions

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a partially differentiable function with respect to every variable x_j, $j = \overline{1,n}$ on A.

SECOND ORDER PARTIAL DERIVATIVES:

• f is **two times partially differentiable** at a with respect to every variable if all partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ are partially differentiable at $a \in A$ with respect to every variable x_k.

The partial derivative with respect to the variable x_k of the partial derivative $\dfrac{\partial f_i}{\partial x_j}$ will be denoted by $\dfrac{\partial^2 f_i}{\partial x_k \partial x_j}(a)$, i.e. $\dfrac{\partial}{\partial x_k}(\dfrac{\partial f_i}{\partial x_j})(a) = \dfrac{\partial^2 f_i}{\partial x_k \partial x_j}(a)$ and will be called **second order partial derivative of** f.

SECOND ORDER FRÉCHET DERIVATIVE:

• f is **two times differentiable** at the point $a \in A$ if the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ are differentiable at a.

The **second order Fréchet derivative** of f at the point a is the function $d_a^2 f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ by the formula $d_a^2 f(u)(v) = \displaystyle\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(a) \cdot u_j \cdot v_k \right) e_i$ where $u, v \in \mathbb{R}^n$, $e_i = (0, ..., 0, 1, 0, ..., 0)$, $i = \overline{1,n}$.

The second order Fréchet derivative of f at a satisfies $\displaystyle\lim_{u \to 0} \frac{\|d_{a+u} f(v) - d_a f(v) - d_a^2 f(u)(v)\|}{\|u\|} = 0$ for every $v \in \mathbb{R}^n$.

Mixed derivative theorem of Schwarz:
If the function f is twice differentiable at a, then $\dfrac{\partial^2 f_i}{\partial x_j \partial x_k}(a) = \dfrac{\partial^2 f_i}{\partial x_k \partial x_j}(a)$ for any $i = \overline{1,m}$ and $j, k = \overline{1,n}$.

Criterion for second order differentiability:
If the second order partial derivatives $\dfrac{\partial^2 f_i}{\partial x_j \partial x_k}$ exist in a neighborhood of a and they are continuous at a, then f is two times differentiable at a.

HIGHER ORDER PARTIAL DERIVATIVES:

• f is **k-times partially differentiable** at $a \in A$ with respect to every variable if f is $(k-1)$-times partially differentiable with respect to every variable on an open neighborhood of a and every $(k-1)$-th order partial derivative $\dfrac{\partial^{k-1} f_i}{\partial x_{j_{k-1}} \cdots \partial x_{j_1}}$ is partially differentiable with respect to every variable x_{j_k} at a.

The **k-th order partial derivative of** f at a is $\dfrac{\partial^k f_i}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}(a) = \dfrac{\partial}{\partial x_{j_k}}(\dfrac{\partial^{k-1} f_i}{\partial x_{j_{k-1}} \cdots \partial x_{j_1}})(a)$.

HIGHER ORDER DIFFERENTIABILITY:

• f is **k-times differentiable** at a if the partial derivatives of order $k-1$, $\dfrac{\partial^{k-1} f_i}{\partial x_{j_{k-1}} \cdots \partial x_{j_1}}$ are differentiable at a.

The **Fréchet derivative of order** k of f at a is the function $d_a^k f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^m$ given by

$$d_a^k f(u^1)(u^2) \cdots (u^k) = \sum_{i=1}^{m} \left(\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\partial^k f_i}{\partial x_{j_k} \cdots \partial x_{j_1}}(a) \cdot u_{j_1}^1 u_{j_2}^2 \cdots u_{j_k}^k \right) e_i$$

The Fréchet derivative of order k of f at a satisfies:

$$\lim_{\|u^k\| \to 0} \frac{\|d_{a+u^k}^{k-1} f(u^1)(u^2) \cdots (u^{k-1}) - d_a^{k-1} f(u^1)(u^2) \cdots (u^{k-1}) - d_a^k f(u^1)(u^2) \cdots (u^k)\|}{\|u^k\|} = 0$$

Mixed derivative theorem:
If the function is k-times differentiable at a, then the following relations hold:

$$\frac{\partial^k f_i}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}}(a) = \frac{\partial^k f_i}{\partial x_{\sigma(j_1)} \partial x_{\sigma(j_2)} \cdots \partial x_{\sigma(j_k)}}(a)$$

Criterion for k-times differentiability:
If the partial derivatives of k-th order exist in a neighborhood of a and they are continuous at a, then f is k-times differentiable at a.

FINDING LOCAL EXTREMA

The point $a \in A$ is a **local minimum point** of the function $f : A \subset \mathbb{R}^n \to R^1$ if there exists a neighborhood $V \subset A$ of a such that $f(a) \leq f(x)$ for any $x \in V$.
The point $a \in A$ is a **global minimum point** of the function $f : A \subset \mathbb{R}^n \to R^1$ if $f(a) \leq f(x)$ for any $x \in A$.
The point $a \in A$ is a **local maximum point** of the function $f : A \subset \mathbb{R}^n \to R^1$ if there exists a neighborhood $V \subset A$ of a such that $f(a) \geq f(x)$ for any $x \in V$.
The point $a \in A$ is a **global maximum point** of the function $f : A \subset \mathbb{R}^n \to R^1$ if $f(a) \geq f(x)$ for any $x \in A$.

Necessary condition for local extrema:
If the function $f : A \subset \mathbb{R}^n \to R^1$ attains a local minimum or maximum value at the point $a \in A$ and all partial derivatives of f exist at a, then $\nabla f(a) = 0$ (i.e. a is a **critical point** of f).

Sufficient condition for local extrema:
Assume that $f : A \subset \mathbb{R}^n \to R^1$ has continuous second order partial derivatives on A and a is a critical point of f.

i) If $d_a^2 f(h)(h) \geq 0$ for $h \in \mathbb{R}^n$ and $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right) \neq 0$, then f has a local minimum at a;

ii) If $d_a^2 f(h)(h) \leq 0$ for $h \in \mathbb{R}^n$ and $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right) \neq 0$, then f has a local maximum at a.

Second partial derivative test for functions of two variables:
Consider the function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ and $a = (a_1, a_2) \in A$ a critical point of f. Consider the Hessian matrix:

$$H_{(a_1,a_2)} f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a_1, a_2) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a_1, a_2) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a_1, a_2) & \frac{\partial^2 f}{\partial x_2^2}(a_1, a_2) \end{pmatrix}$$

Consider the principal minors of the Hessian matrix:

$$\Delta_1 = \frac{\partial^2 f}{\partial x_1^2}(a_1, a_2) \qquad \text{and} \qquad \Delta_2 = \det\left(H_{(a_1,a_2)} f\right)$$

- if $\Delta_1 > 0$ and $\Delta_2 > 0$ then $a = (a_1, a_2)$ is a **local minimum point** of f;
- if $\Delta_1 < 0$ and $\Delta_2 > 0$ then $a = (a_1, a_2)$ is a **local maximum point** of f;
- if $\Delta_2 < 0$ then $a = (a_1, a_2)$ is a **saddle point** of f;
- if $\Delta_2 = 0$ then this test is inconclusive.

LAGRANGE MULTIPLIERS AND CONSTRAINED OPTIMIZATION

Consider a function $f : A \subset \mathbb{R}^n \to \mathbb{R}^1$, where A is an open set and the set $\Gamma \subset A$, defined by:

$$\Gamma = \{x \in A \,|\, g_i(x) = 0, \quad i = \overline{1,p}\} \qquad \text{where } g_i : A \to \mathbb{R}^1 \text{ and } p < n$$

If the restriction $f|_\Gamma$ has an extreme point $a \in \Gamma$, then this is called **conditional extreme point**.

Assume that f and g_i, $i = \overline{1,p}$ are continuously differentiable near the conditional extreme point $a \in \Gamma$ and $\nabla g_i(a)$, $i = \overline{1,p}$ are linearly independent vectors of $\mathbb{R}^n$. Then there exist some constants $\lambda_1, \lambda_2, ..., \lambda_p$ such that

$$\nabla f(a) = \sum_{i=1}^{p} \lambda_i \nabla g_i(a)$$

Example: Two variables and one constraint:
If we want to maximize (minimize) the function $f : A \subset \mathbb{R}^2 \to \mathbb{R}^1$ subject to the constraint $g(x, y) = 0$, we first need to solve the system of three equations

$$\begin{cases} g(x, y) = 0 \\ \frac{\partial f}{\partial x}(x, y) = \lambda \frac{\partial g}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) = \lambda \frac{\partial g}{\partial y}(x, y) \end{cases}$$

with respect to the variables x, y, λ. The points (x, y) that we find are the only possible locations of the extrema of f subject to the constraint $g(x, y) = 0$.

Ex.1 Compute the first and second order Fréchet derivatives for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{xy} + 2x^3y^2 + y - 3xy$ at $a = (0,2)$.

Solution: Compute the first order partial derivatives of the function f.

$$\frac{\partial f}{\partial x} = (e^{xy} + 2x^3y^2 + y - 3xy)'_x = e^{xy} \cdot (xy)'_x + 2y^2 \cdot (x^3)'_x + (y)'_x - 3y \cdot (x)'_x = ye^{xy} + 6x^2y^2 - 3y$$

$$\frac{\partial f}{\partial y} = (e^{xy} + 2x^3y^2 + y - 3xy)'_y = e^{xy} \cdot (xy)'_y + 2x^3 \cdot (y^2)'_y + (y)'_y - 3x \cdot (y)'_y = xe^{xy} + 4x^3y + 1 - 3x$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,2) = 2 \cdot e^{0\cdot 2} + 6 \cdot 0^2 \cdot 2^2 - 3 \cdot 2 = -4 \text{ and } \frac{\partial f}{\partial y}(0,2) = 0 \cdot e^{0\cdot 2} + 4 \cdot 0^3 \cdot 2 + 1 - 3 \cdot 0 = 1$$

The first order Fréchet derivative of the function f at the point $a = (0,2)$ is:

$d_{(0,2)}f(h_1,h_2) = \nabla f(0,2) \cdot h = (-4,1) \cdot (h_1,h_2) = -4h_1 + h_2$

Compute the second order partial derivatives of the function f.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (ye^{xy} + 6x^2y^2 - 3y)'_x = ye^{xy} \cdot (xy)'_x + 6y^2 \cdot (x^2)'_x - (3y)'_x = y^2e^{xy} + 12xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (ye^{xy} + 6x^2y^2 - 3y)'_y = (y)'_y \cdot e^{xy} + y \cdot e^{xy} \cdot (xy)'_y + 6x^2 \cdot (y^2)'_y - 3 \cdot (y)'_y = e^{xy} + xye^{xy} + 12x^2y - 3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (xe^{xy} + 4x^3y + 1 - 3x)'_x = (x)'_x \cdot e^{xy} + x \cdot e^{xy} \cdot (xy)'_x + 4y \cdot (x^3)'_x - 3 \cdot (y)'_x = e^{xy} + xye^{xy} + 12x^2y - 3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (xe^{xy} + 4x^3y + 1 - 3x)'_y = xe^{xy} \cdot (xy)'_y + 4x^3 \cdot (y)'_y - (3x)'_y = x^2e^{xy} + 4x^3$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(0,2) = 4,\ \frac{\partial^2 f}{\partial x \partial y}(0,2) = \frac{\partial^2 f}{\partial y \partial x} = -2 \text{ and } \frac{\partial^2 f}{\partial y^2}(0,2) = 0 \text{ (check!)}$$

The second order Fréchet derivative of the function f at the point $a = (0,2)$ is:

$$\begin{aligned} d^2_{(0,2)}f(u_1,u_2)(v_1,v_2) &= \frac{\partial^2 f}{\partial x^2}(0,2)u_1v_1 + \frac{\partial^2 f}{\partial x \partial y}(0,2)u_1v_2 + \frac{\partial^2 f}{\partial y \partial x}u_2v_1 + \frac{\partial^2 f}{\partial y^2}(0,2)u_2v_2 \\ &= 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 0 \cdot u_2v_2 \\ &= 4u_1v_1 - 2(u_1v_2 + u_2v_1) \end{aligned}$$

Ex.2 Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Solution: Compute the first order partial derivatives of the function f.

$$\frac{\partial f}{\partial x} = (3x^2y + y^3 - 3x^2 - 3y^2 + 2)'x = 3y \cdot (x^2)'_x + (y^3)'_x - 3 \cdot (x^2)'_x - (3y^2)'_x + 2'_x = 6xy - 6x$$

$$\frac{\partial f}{\partial y} = (3x^2y + y^3 - 3x^2 - 3y^2 + 2)'_y = 3x^2 \cdot (y)'_y + (y^3)'_y - (3x^2)'_y - 3 \cdot (y^2)'_y + 2'_y = 3x^2 + 3y^2 - 6y$$

Find the critical points of the function f. Solve the system:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 6xy - 6x & = 0 \\ 3x^2 + 3y^2 - 6y & = 0 \end{cases} \qquad 6xy - 6x = 0 \Leftrightarrow 6x(y-1) = 0 \Leftrightarrow x = 0 \text{ or } y = 1$$

$x = 0 \Rightarrow 3y^2 - 6y = 0 \Leftrightarrow 3y(y-2) = 0 \Leftrightarrow y = 0$ or $y = 2$

$y = 1 \Rightarrow 3x^2 + 3 - 6 = 0 \Leftrightarrow 3x^3 = 3 \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1$ or $x = -1$

The critical points are $(0,0)$, $(0,2)$, $(1,1)$ şi $(-1,1)$.

Compute the Hessian matrix at each critical point.

$$Hf = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} (6xy - 6x)'_x & (6xy - 6x)'_y \\ (3x^2 + 3y^2 - 6y)'_x & (3x^2 + 3y^2 - 6y)'_y \end{pmatrix} = \begin{pmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{pmatrix}$$

$H_{(0,0)}f = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$ $\quad \Delta_1 = -6 < 0$, $\Delta_2 = \det(H_{(0,0)}f) = 36 > 0 \Rightarrow (0,0)$-local maximum point

$H_{(0,2)}f = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ $\quad \Delta_1 = 6 > 0$, $\Delta_2 = H_{(0,2)}f = 36 > 0 \Rightarrow (0,2)$-local minimum point

$H_{(1,1)}f = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$ $\quad \Delta_1 = 0$, $\Delta_2 = \det(H_{(1,1)}f) = -36 < 0 \Rightarrow (1,1)$-saddle point

$H_{(-1,1)}f = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \quad \Delta_1 = 0,\ \Delta_2 = \det(H_{(-1,1)}f) = -36 < 0 \Rightarrow (-1,1)$-saddle point

Ex.3 Find the maximum and minimum values of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 4x^2 + 10y^2$ subject to the constraint $x^2 + y^2 = 4$.

Solution: Let $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x^2 + y^2 - 4$. Solve the system:

$$\begin{cases} g(x,y) = 0 \\ \dfrac{\partial f}{\partial x}(x,y) = \lambda \dfrac{\partial g}{\partial x}(x,y) \\ \dfrac{\partial f}{\partial y}(x,y) = \lambda \dfrac{\partial g}{\partial y}(x,y) \end{cases} \qquad \begin{matrix} \dfrac{\partial f}{\partial x}(x,y) = (4x^2 + 10y^2)'_x = 8x & \dfrac{\partial g}{\partial x}(x,y) = (x^2 + y^2 - 4)'_x = 2x \\ \dfrac{\partial f}{\partial x}(x,y) = (4x^2 + 10y^2)'_y = 20y & \dfrac{\partial g}{\partial y}(x,y) = (x^2 + y^2 - 4)'_y = 2y \end{matrix}$$

The system becomes

$$\begin{cases} x^2 + y^2 = 4 \\ 8x = 2\lambda x \\ 20y = 2\lambda y \end{cases}$$

From the second equation, we have: $8x - 2\lambda x = 0 \Leftrightarrow 2x(4-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 4$

If $x = 0$, by substituting in the first equation of the system, we obtain

$y^2 = 4 \Leftrightarrow y = \pm 2 \Rightarrow (0,2)$ and $(0,-2)$ are critical points.

If $\lambda = 4$, by plugging in the third equation of the system, we have

$20y = 8y \Leftrightarrow 12y = 0 \Leftrightarrow y = 0$

Then, replacing $y = 0$ into the first equation of the system, it results that

$x^2 = 4 \Leftrightarrow x = \pm 2 \Rightarrow (2,0)$ and $(-2,0)$ are critical points.

Compute the values of the funtion f at the critical points previously found.

$f(0,2) = f(0,-2) = 40$ and $f(2,0) = f(-2,0) = 16$.

Thus, $f_{min} = 16$ at the points $(2,0)$ and $(-2,0)$ and $f_{max} = 40$ at the points $(0,2)$ and $(0,-2)$.

CALCULUS HANDOUT 10
HIGHER ORDER DIFFERENTIABILITY. LOCAL EXTREMA - exercises

1. Compute the first and second order Fréchet derivatives for the following functions:
   1. $f(x,y) = x^3 + 8y^3 - 6xy - 1$
   2. $f(x,y,z) = e^{ax+by+cz}$
   3. $f(x,y) = \ln(x+y)$
   4. $f(x,y) = \sin(x+3y)$
   5. $f(x,y) = e^{x^2+y^2}$
   6. $f(x,y,z) = xyz$
   7. $f(x_1,x_2,...,x_p) = \ln(x_1+a_1)(x_2+a_2)...(x_p+a_p)$
   8. $f(x,y,z) = \cos xyz$

2. Find and classify the critical points of the following functions of two variables:
   1. $f(x,y) = 2x^2 + y^2 + 4x - 4y + 5$
   2. $f(x,y) = 10 + 12x - 12y - 3x^2 - 2y^2$
   3. $f(x,y) = 2x^2 - 3y^2 + 2x - 3y + 7$
   4. $f(x,y) = xy + 3x - 2y + 4$
   5. $f(x,y) = 2x^2 + 2xy + y^2 + 4x - 2y + 1$
   6. $f(x,y) = x^2 + 4xy + 2y^2 + 4x - 8y + 3$
   7. $f(x,y) = x^3 + y^3 + 3xy + 3$
   8. $f(x,y) = x^2 - 2xy + y^3 - y$
   9. $f(x,y) = x^3 + 8y^3 - 6xy - 1$
   10. $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$
   11. $f(x,y) = x^3 + 8y^3 - 6xy + 1$
   12. $f(x,y) = x^3 + 3xy^2 - 15x - 12y$
   13. $f(x,y) = xy - 2x - 2y - x^2 - y^2$
   14. $f(x,y) = x^2 + y^4 + 2xy$
   15. $f(x,y) = 2 - x^4 + 2x^2 - y^2$
   16. $f(x,y) = x^3 - 3x + 3xy^2$
   17. $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$
   18. $f(x,y) = x^2 - 2y^2 + xy - 2x - y + 5$
   19. $f(x,y) = 12x + 25y - 2x^2 - 6y^2$
   20. $f(x,y) = x^3 + 2y^2 - 27x - 8y$
   21. $f(x,y) = 2(x-y)^2 - x^4 - y^4$
   22. $f(x,y) = x^2 + y^2 - xy + x + y$
   23. $f(x,y) = x^3 + 6xy + 3y^2 - 9x$
   24. $f(x,y) = 4xy - 2x^4 - y^2$
   25. $f(x,y) = (x\cos y - y\sin y)\, e^x$
   26. $f(x,y) = \ln(x^2 + y^2)$
   27. $f(x,y) = e^{\frac{x}{2}}\,(x + y^2)$
   28. $f(x,y) = x^2 + 2y^2 + \pi\cos x\,\cos y$
   29. $f(x,y) = e^{x^2+y^2} - x^2 - 2y^2$
   30. $f(x,y) = (1-x)(1-y)(x+y-1)$
   31. $f(x,y,z) = x^3 + y62 + xz^2 + 12xy + 2z$
   32. $f(x,y,z) = x^3 - y^2 - z^2 - 12x + 12y + 14z$
   33. $f(x,y,z) = x^2 + 3y^2 + 2x^2 - 2xy + 2xz$
   34. $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$

3. Find the maximum and minimum values (if any) of the following functions subject to the following constraint:
   1. $f(x,y) = 2x + y;\ x^2 + y^2 = 1$
   2. $f(x,y) = x + y;\ x^2 + 4y^2 = 1$
   3. $f(x,y) = x^2 - y^2;\ x^2 + y^2 = 4$
   4. $f(x,y) = x^2 + y^2;\ 2x + 3y = 6$
   5. $f(x,y) = xy;\ 4x^2 + 9y^2 = 36$
   6. $f(x,y) = 4x^2 + 9y^2;\ x^2 + y^2 = 1$
   7. $f(x,y) = 3x + y;\ x^2 + y^2 = 10$
   8. $f(x,y) = x^2 + y^2 + 4x - 4y;\ x^2 + y^2 = 9$
   9. $f(x,y) = x^2 + 2y^2;\ x^2 + y^2 = 1$
   10. $f(x,y) = 2x^2 + 3y^2 - 4x - 5;\ x^2 + y^2 = 16$
   11. $f(x,y,z) = x^2 + y^2 + z^2;\ 3x + 2y + z = 6$
   12. $f(x,y,z) = 3x + 2y + z;\ x^2 + y^2 + z^2 = 1$
   13. $f(x,y,z) = x + y + z;\ x^2 + 4y^2 + 9z^2 = 36$
   14. $f(x,y,z) = xyz;\ x^2 + y^2 + z^2 = 1$
   15. $f(x,y,z) = xy + 2z;\ x^2 + y^2 + z^2 = 36$
   16. $f(x,y,z) = x^2y^2z^2;\ x^2 + 4y^2 + 9z^2 = 27$

Extra exercises

4. Find the points of the parabola $y = (x-1)^2$ that are closest to the origin.

5. Find the points of the ellipse $4x^2 + 9y^2 = 36$ that are closest to and farthest from the point $(3,2)$.

6. Consider a right triangle of hypothenuse z and legs x and y, and fixed perimeter P. Maximize its area $A = \dfrac{xy}{2}$ subject to the constraints $x + y + z = P$ and $x^2 + y^2 = z^2$. In particular, show that the optimal such triangle is isosceles ($x = y$).