Algorithms and Data Structures (II)

Gabriel Istrate

March 18, 2020

- Wrap up hash tables.
- Skip lists.
- Binary search trees
- Randomized binary search trees

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- *Implementation (so far)*
  - direct access tables. Linked lists. Hash tables. Skip Lists. Binary Search Trees.

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  - *iteration:* **TREE-SUCCESSOR**(x) and **TREE-PREDECESSOR**(x) find the successor and predecessor, respectively, of an element x

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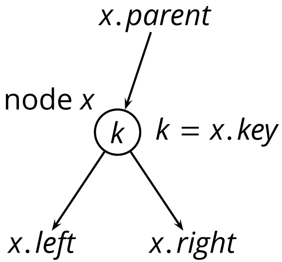
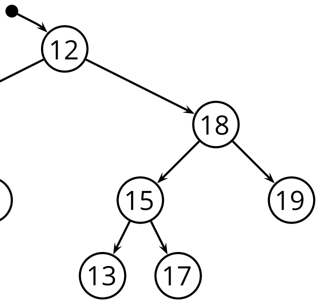
Node x

- $x.parent$ is the parent of node x
- $x.key$ is the key stored in node x
- $x.left$ is the left child of node x
- $x.right$ is the right child of node x



Binary Search Trees (3)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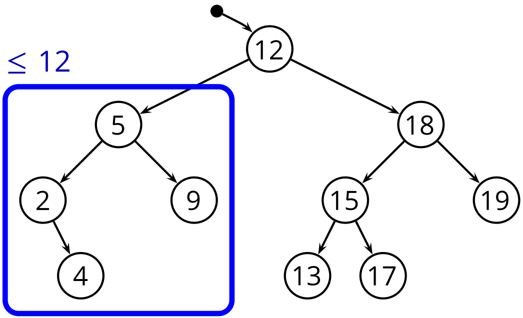
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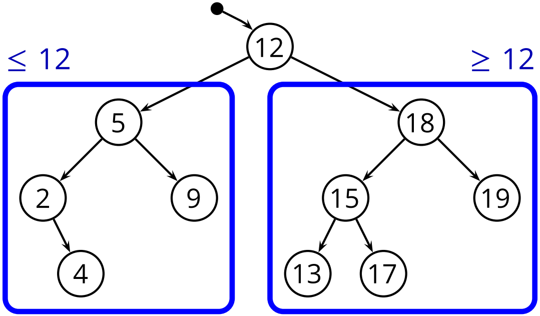

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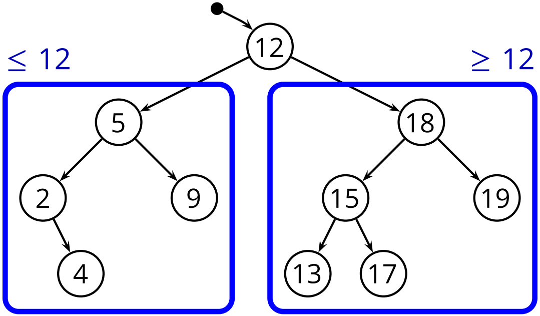

- ***Binary-search-tree property***
  - for all nodes x, y, and z
  - $y \in \textit{left-subtree}(x) \Rightarrow y.\textit{key} \leq x.\textit{key}$
  - $z \in \textit{right-subtree}(x) \Rightarrow z.\textit{key} \geq x.\textit{key}$

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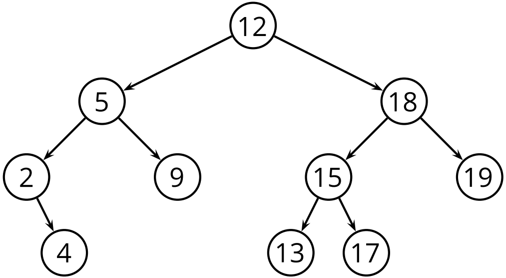

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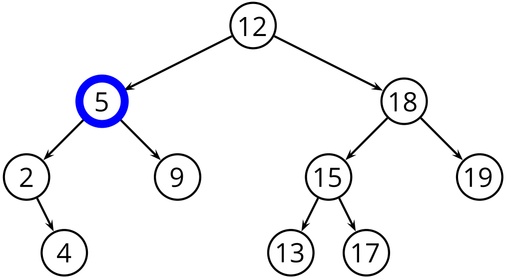

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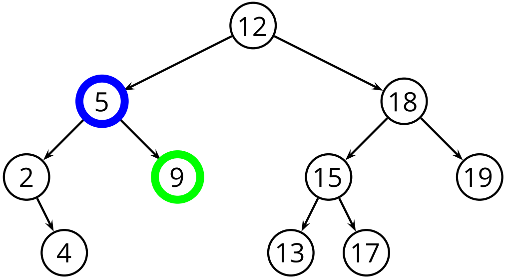

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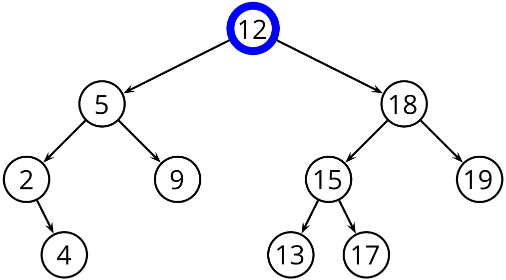

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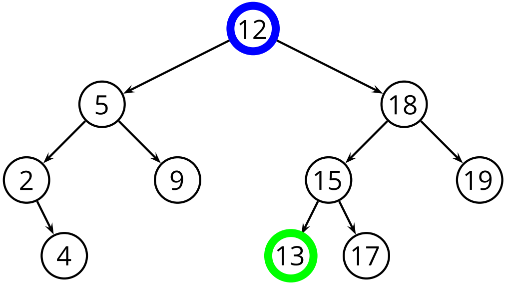

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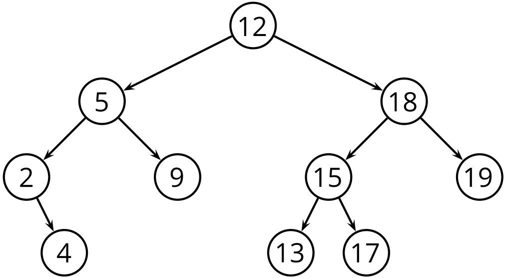

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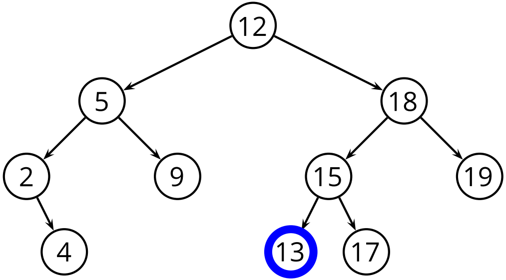

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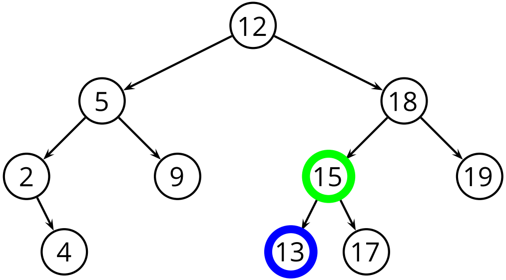

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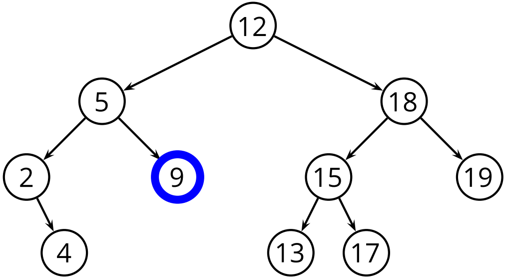

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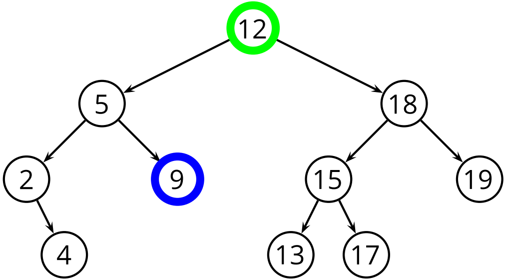

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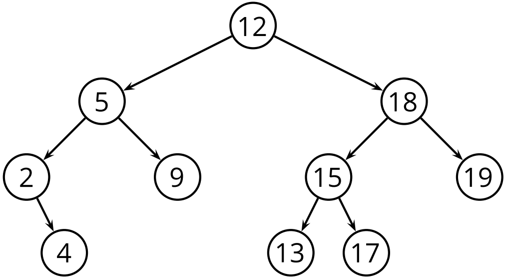

- The successor of *x* is the *minimum* of the *right* subtree of *x*, if that exists
- Otherwise it is the *first ancestor a* of *x* such that *x* falls in the *left* subtree of *a*

Successor and Predecessor(2)

```
TREE-SUCCESSOR(x)
if x.right ≠ NIL
    return TREE-MINIMUM(x.right)
y = x.parent
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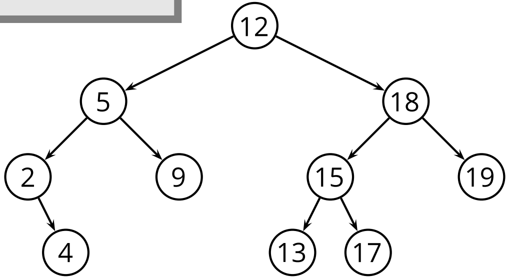

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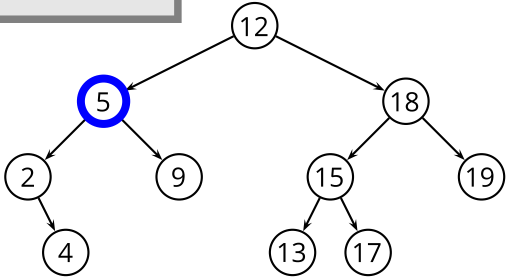

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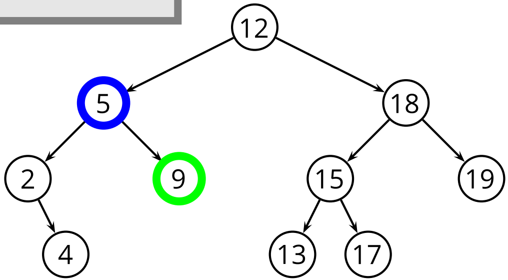

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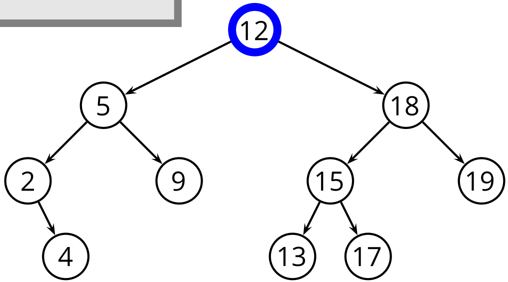

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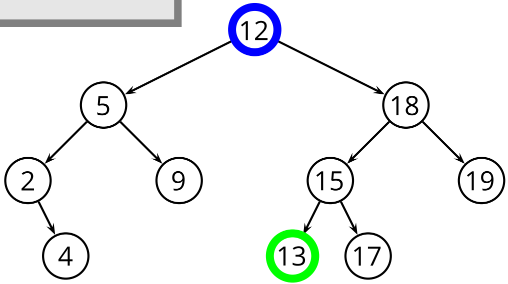

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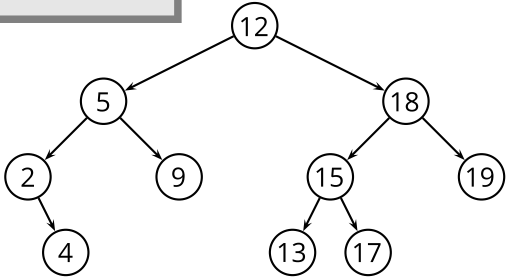

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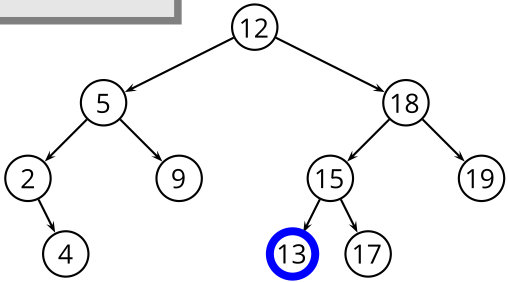

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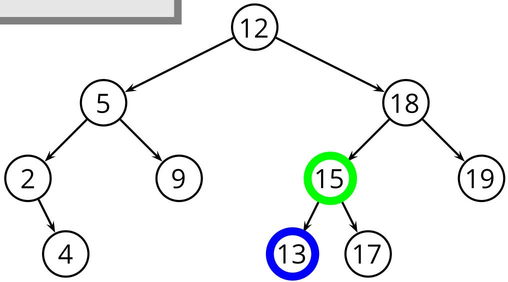

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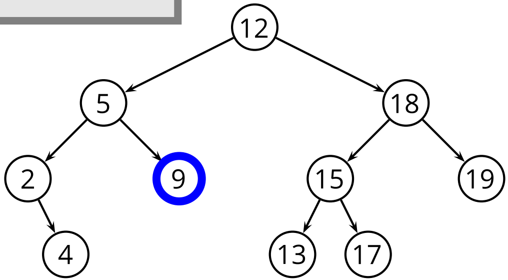

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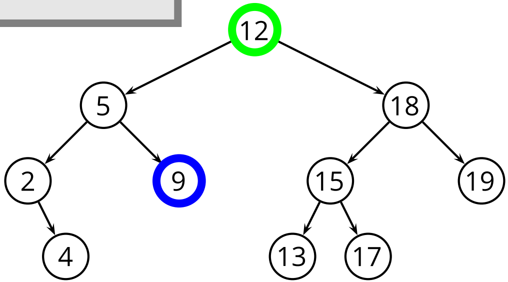

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$$T(n) = O(n)$$

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ITERATIVE-TREE-SEARCH(T, k)
  x = T.root
  while x ≠ NIL ∧ k ≠ x.key
      if k < x.key
          x = x.left
      else x = x.right
  return x
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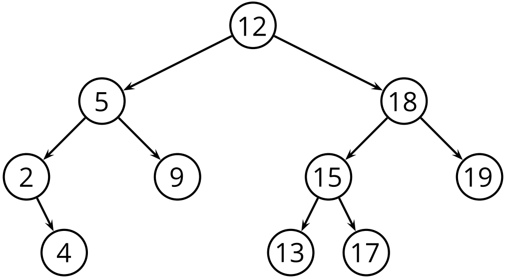
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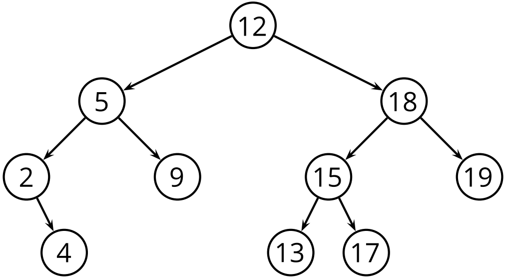

- *Idea*
  - in order to insert x, we *search* for x (more precisely $x.key$)
  - if we don't find it, we add it where the search stopped

Insertion (2)

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y = NIL
x = T.root
while x ≠ NIL
    y = x
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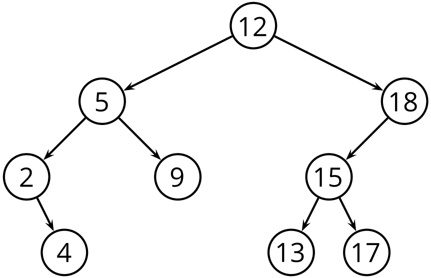

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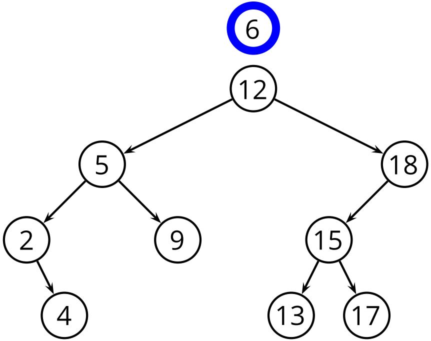

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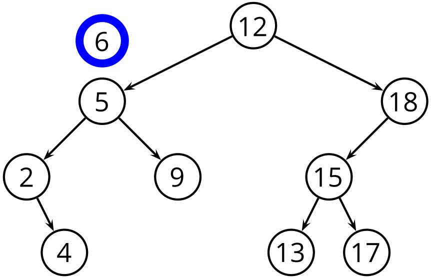

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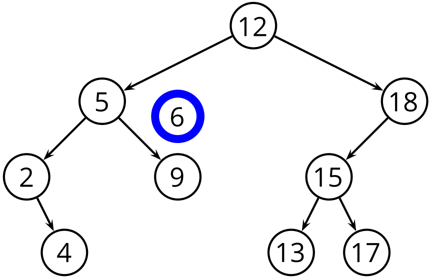

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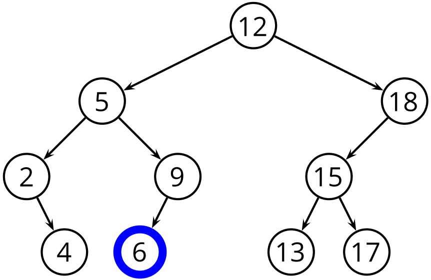

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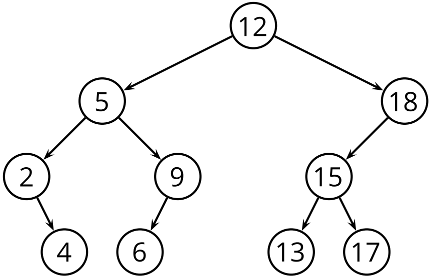

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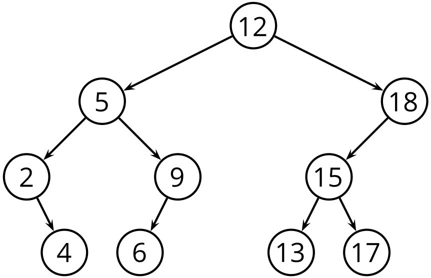

$$T(n) = \Theta(h)$$

- Both insertion and search operations have complexity h, where h is the height of the tree

Observation

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- *Idea:* use randomization to turn all cases in the average case

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- *Idea 2:* we can obtain a random permutation of the input sequence by randomly alternating two insertion procedures
  - *tail insertion:* this is what **TREE-INSERT** does
  - *head insertion:* for this we need a new procedure **TREE-ROOT-INSERT**
    - inserts n in T as if n was inserted as the first element

```
TREE-RANDOMIZED-INSERT1(T, z)
  r = uniformly rand. val. from {1, ..., t.size + 1}
  if r = 1
        TREE-ROOT-INSERT(T, z)
  else TREE-INSERT(T, z)
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- It is true that any node has the same probability of being inserted at the top

Randomized Insertion (2)

TREE-RANDOMIZED-INSERT1(T, z)

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r = uniformly rand. val. from {1, ..., t.size + 1}
if r = 1
      TREE-ROOT-INSERT(T, z)
else TREE-INSERT(T, z)
```

- Does this really simulate a random permutation?
  - i.e., with all permutations being equally likely?
  - no, clearly the last element can only go to the top or to the bottom

- It is true that any node has the same probability of being inserted at the top
  - this suggests a recursive application of this same procedure

```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
        return z
r = uniformly random value from {1, ..., t.size + 1
if r = 1                                // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

```
TREE-RANDOMIZED-INSERT(t, z)
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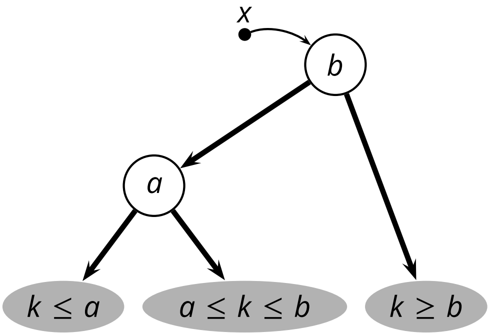
r = uniformly random value from {1, . . . , t.size + 1
if r = 1                                  // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

- Looks like this one really simulates a random permutation...


x
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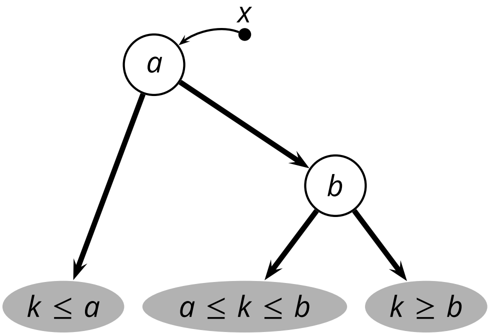
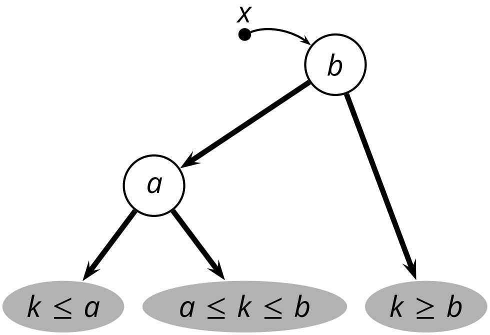
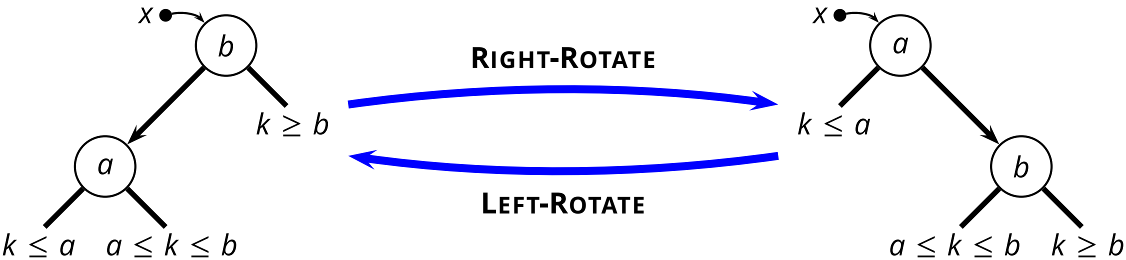
a
k ≤ a
a ≤ k ≤ b
k ≥ b



- x = **Right-Rotate**(x)

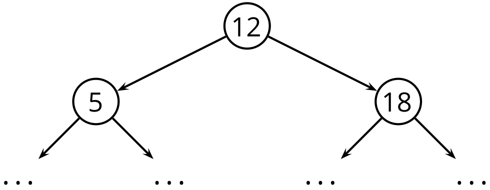


- x = **RIGHT-ROTATE**(x)
- x = **LEFT-ROTATE**(x)




```
RIGHT-ROTATE(x)
    l = x.left
    x.left = l.right
    l.right = x
    return l
```

```
LEFT-ROTATE(x)
    r = x.right
    x.right = r.left
    r.left = x
    return r
```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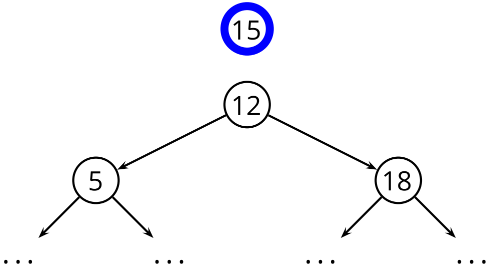
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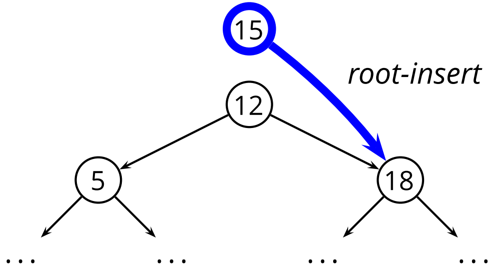
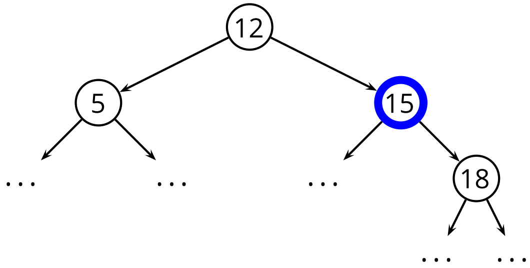


1. Recursively insert *z* at the root of the appropriate subtree (right)



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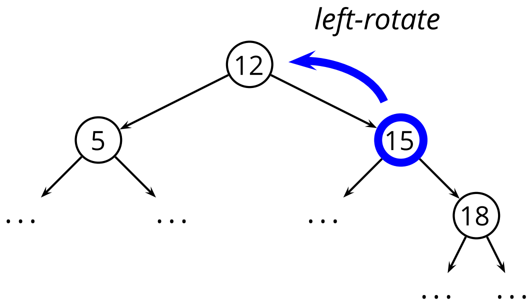
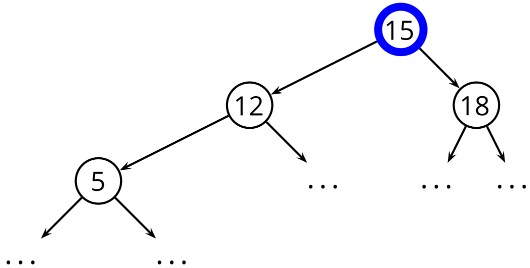


1. Recursively insert z at the root of the appropriate subtree (right)
2. Rotate x with z (left-rotate)



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```
TREE-ROOT-INSERT(x, z)
if x = NIL
        return z
if z.key < x.key
        x.left = TREE-ROOT-INSERT(x.left, z)
        return RIGHT-ROTATE(x)
else x.right = TREE-ROOT-INSERT(x.right, z)
        return LEFT-ROTATE(x)
```

- General strategies to deal with complexity in the worst case

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  - *randomization:* turns any case into the average case
    - the worst case is still possible, but it is extremely improbable

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    - relatively expensive but "amortized" operations

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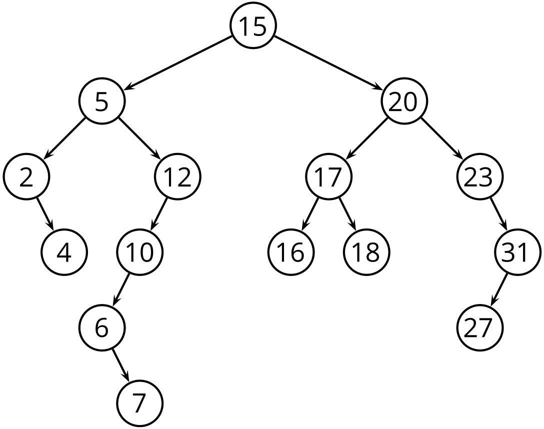
  - *randomization:* turns any case into the average case
    - the worst case is still possible, but it is extremely improbable
  - *amortized maintenance:* e.g., balancing a BST or resizing a hash table
    - relatively expensive but “amortized” operations
  - *optimized data structures:* a self-balanced data structure
    - guaranteed $O(\log n)$ complexity bounds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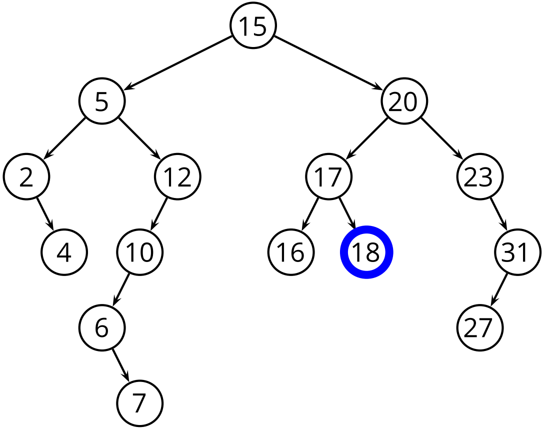
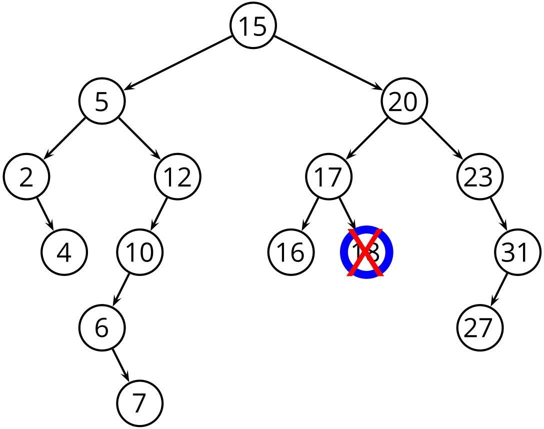
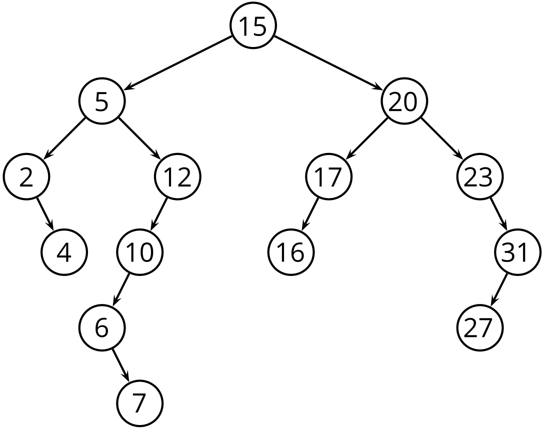


1. z has no children



1. *z* has no children
   - simply remove *z*



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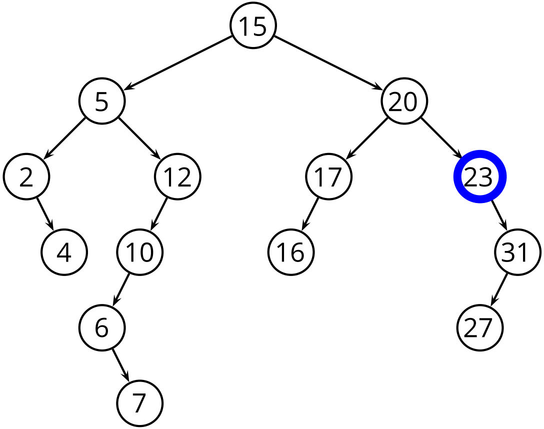

1. z has no children
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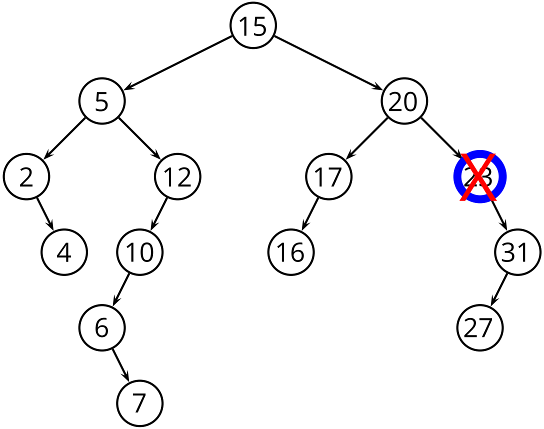

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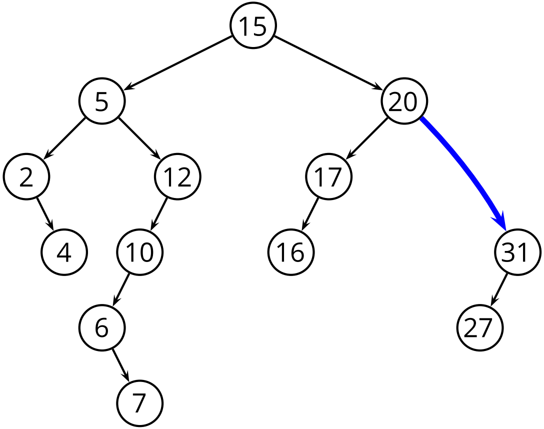

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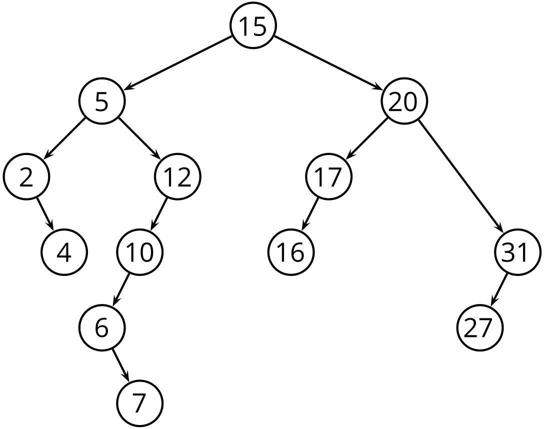
2. *z* has one child
   - remove *z*
   - connect $z.parent$ to $z.right$



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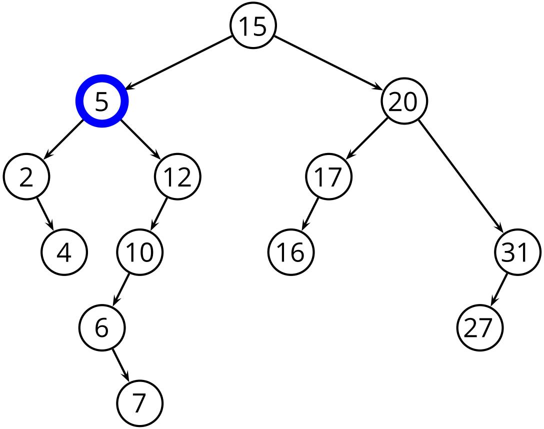

1. *z* has no children
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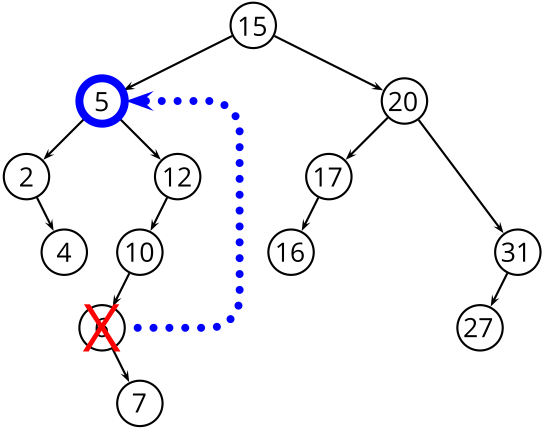
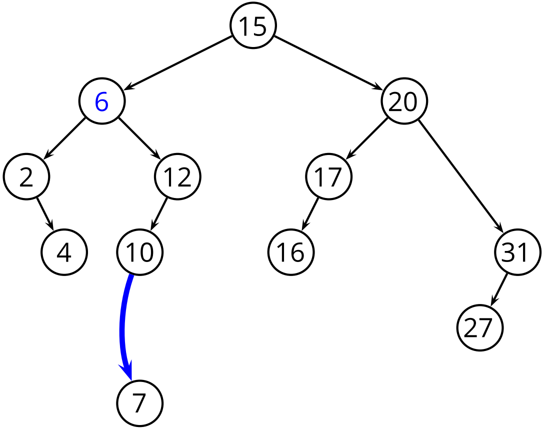
2. *z* has one child
   - remove *z*
   - connect *z.parent* to *z.right*
3. *z* has two children



1. *z* has no children
   - simply remove *z*
2. *z* has one child
   - remove *z*
   - connect $z.parent$ to $z.right$
3. *z* has two children
   - replace *z* with $y = \textbf{Tree-Successor}(z)$
   - remove *y* (1 child!)



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   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$
3. z has two children
   - replace z with $y = \textsc{Tree-Successor}(z)$
   - remove y (1 child!)
   - connect $y.parent$ to $y.right$

Deletion (2)

```
TREE-DELETE(T, z)
    if z.left = NIL or z.right = NIL
        y = z
    else y = TREE-SUCCESSOR(z)
    if y.left ≠ NIL
        x = y.left
    else x = y.right
    if x ≠ NIL
        x.parent = y.parent
    if y.parent == NIL
        T.root = x
    else if y = y.parent.left
            y.parent.left = x
        else y.parent.right = x
    if y ≠ z
        z.key = y.key
        copy any other data from y into z
```