Algorithms and Data Structures (II)

Gabriel Istrate

March 18, 2020

- Wrap up hash tables.
- Skip lists.
- Binary search trees
- Randomized binary search trees

Where are we ?

- A *dictionary* is an abstract data structure that represents a set of elements (or keys)
  - a *dynamic set*

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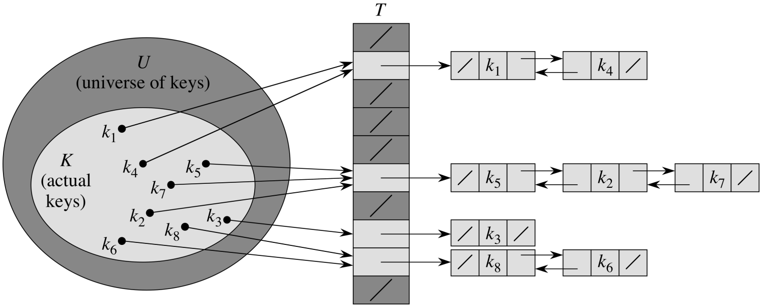
- *Implementation (so far)*
  - direct access tables. Linked lists. Hash tables.


T
U
(universe of keys)
K
(actual
keys)
k_1
k_4
k_5
k_7
k_2
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k_1
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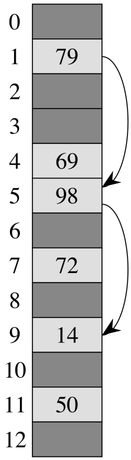

Three versions: linear/quadratic/double probing.

Algorithm	***Average** Complexity (Search successful/not)*	
INSERT/SEARCH/DELETE, CHAINING:	$O(1+\alpha)$	✓
SEARCH, LINEAR PROBING:	$\frac{1}{2}(1+\frac{1}{1-\alpha}), \frac{1}{2}(1+\frac{1}{1-\alpha^2})$	✓
SEARCH, QUADRATIC PROBING:	$1 - ln(1-\alpha) - \frac{\alpha}{2}, \frac{1}{1-\alpha} - \alpha - ln(1-\alpha)$	✓
SEARCH, DOUBLE HASHING:	$\frac{1}{\alpha} ln(1-\alpha), \frac{1}{1-\alpha}$	✓

Reference, probing complexities: Drozdek/Knuth.

- practical ! ✓
- Hard to analyze mathematically: those results under uniform hashing (not at all clear) $\times$
- Somewhat hard to engineer. $\times$.

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In practice $log(n)$ is a small number !

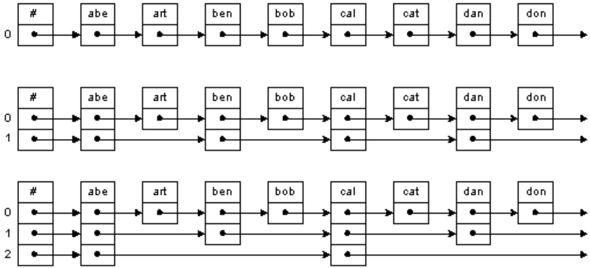
Caution

Topic **not** in Cormen. See Drozdek for details/C++ implementation.

- Problem with linked list: *search is slow !*... even when elements sorted.
- Solution: lists of ordered elements that allow skipping some elements to speed up search.
- Skip lists: variant of ordered linked lists that makes such search possible.

More advanced data structure (W. Pugh "Skip lists: a Probabilistic Alternative to Balanced Trees", Communication of the ACM 33(1990), pp. 668-676.) *If anyone curious/interested in data structures/algorithms, can give paper to read; taste how a research article looks like.*

Skip lists



Where does this ever get applied ? ...

Skip lists in real life

According to Wikipedia:

- MemSQL - skip lists as prime indexing structure for its database technology.
- Cyrus IMAP server - "skiplist" backend DB implementation
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.
- QMap (up to Qt 4) template class of Qt that provides a dictionary.
- Redis, ANSI-C open-source persistent key/value store for Posix systems, skip lists in implementation of ordered sets.
- nessDB, a very fast key-value embedded Database Storage Engine.
- skipdb: open-source DB format using ordered key/value pairs.
- ConcurrentSkipListSet and ConcurrentSkipListMap in the Java 1.6 API.

According to Wikipedia:

- Speed Tables: fast key-value datastore for Tcl that use skiplists for indexes and lockless shared memory.
- leveldb, a fast key-value storage library written at Google that provides an ordered mapping from string keys to string values
- MuQSS Scheduler for the Linux kernel uses skip lists
- SkipMap uses skip lists as base data structure to build a more complex 3D Sparse Grid for Robot Mapping systems.

What we want

$k = 1, \ldots, \lfloor \log_2(n) \rfloor$, $1 \leq i \leq \lfloor n/2^{k-1} \rfloor - 1$.

- Item $2^{k-1} \cdot i$ points to item $2^{k-1} \cdot (i+1)$.
- every second node points to positions two node ahead,
- every fourth node points to positions four nodes ahead,
- every eigth node points to positions eigth nodes ahead,
- $\ldots \ldots$, and so on.

- **Different number of pointers in different nodes in the list !**
- half the nodes only one pointer.
- a quarter of the nodes two pointers,
- an eigth of the nodes four pointers,
- $\ldots \ldots$, and so on.
- $n \log_2(n)/2$ pointers.

1. First follow pointers on the highest level until a larger element is found or the list is exhausted.
2. If a larger element is found, restart search from its predecessor, this time on a lower level.
3. Continue doing this until element found, or you reach the first level and a larger element or the end of the list.

Major problem

- When inserting/deleting a node, pointers of prev/next nodes have to be restructured.
- Solution: rather than equal spacing, random spacing on a level.
- Invariant: Number of nodes on each level: equal, in expectation to what it would be under equal spacing

Principle

If you're traveling 10 meters in 10 steps, a step is on average one meter.

Inserting and deleting nodes (II)

- Level numbering: start with zero.
- New node inserted: probability 1/2 on first level, 1/4 second level, 1/8 third level, . . ., etc.
- Function *chooseLevel*: chooses randomly the level of the new node.
- Generate random number. If in [0,1/2] level 1, [1/2,3/4] level 2, etc.
- To delete node: have to update all links.

Computing the i'th element faster than in $O(i)$

- If we record "step sizes" in our lists we can even mimic indexing !
- Start on highest level.
- If step too big, restart search from predecessor, this time on a lower level.
- Continue doing this until element found.

Update "step sizes" by insertion/deletion

Easy if you have doubly linked lists.

- On deletion: $pred[i].size+ = deleted.size$ on all levels i.
- On insertion: Simply keep track of predecessors and index of the inserteed sequence.

Method	***Average***	***Worst-Case***
SPACE: ✓	$O(n)$	$O(nlog(n))$
SEARCH: ✓	$O(\log(n))$	$O(n)$
INSERT: ✓	$O(\log(n))$	$O(n)$
DELETE: ✓	$O(\log(n))$	$O(n)$

- quite practical ! ✓
- Probabilistic, worst-case still bad. ×
- Not completely easy to implement. ×.

Compared to what ?

Binary search trees. Will learn about them next.

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  - over a ***totally ordered domain***

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  - **TREE-MINIMUM**(T) finds the smallest element in the tree
  - **TREE-MAXIMUM**(T) finds the largest element in the tree
  - *iteration:* **TREE-SUCCESSOR**(x) and **TREE-PREDECESSOR**(x) find the successor and predecessor, respectively, of an element x

- *Implementation*
  - T represents the tree, which consists of a set of ***nodes***

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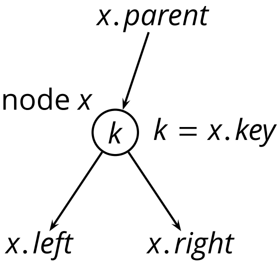
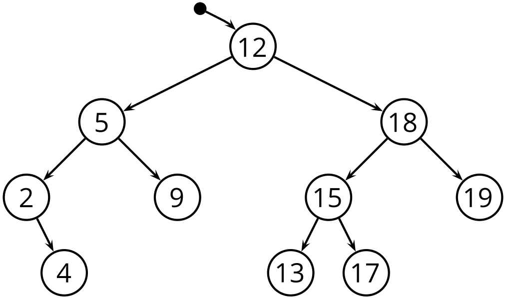
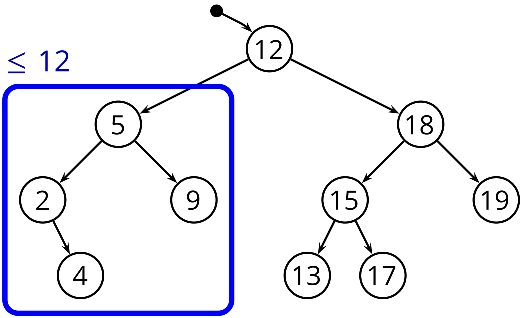
Node x

- $x.parent$ is the parent of node x
- $x.key$ is the key stored in node x
- $x.left$ is the left child of node x
- $x.right$ is the right child of node x



Binary Search Trees (3)




≤ 12
12
5
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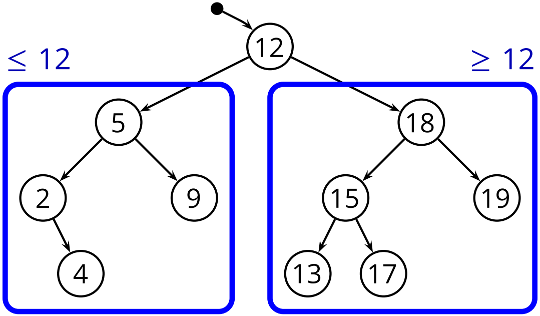
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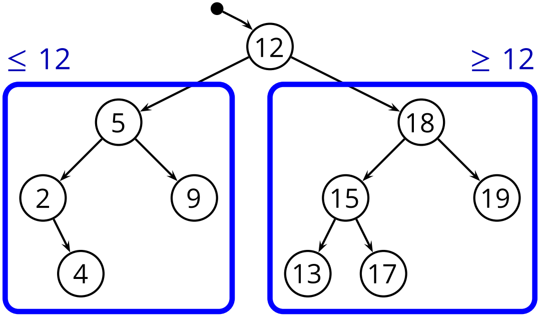

- ***Binary-search-tree property***
  - for all nodes x, y, and z
  - $y \in \textit{left-subtree}(x) \Rightarrow y.key \leq x.key$
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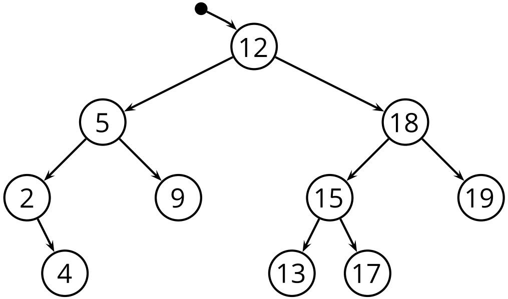
- We want to go through the set of keys *in order*



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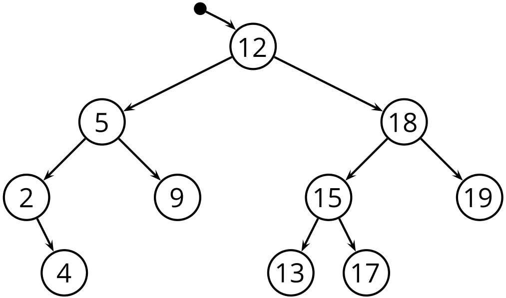

2 4 5 9 12 13 15 17 18 19

- A recursive algorithm

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```
INORDER-TREE-WALK(x)
    if x ≠ NIL
        INORDER-TREE-WALK(x.left)
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```

- A recursive algorithm

```
INORDER-TREE-WALK(x)
1  if x ≠ NIL
2      INORDER-TREE-WALK(x.left)
3      print x.key
4      INORDER-TREE-WALK(x.right)
```

And then we need a “starter” procedure

```
INORDER-TREE-WALK-START(T)
1  INORDER-TREE-WALK(T.root)
```

```
PREORDER-TREE-WALK(x)
    if x ≠ NIL
        print x.key
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Preorder Tree Walk

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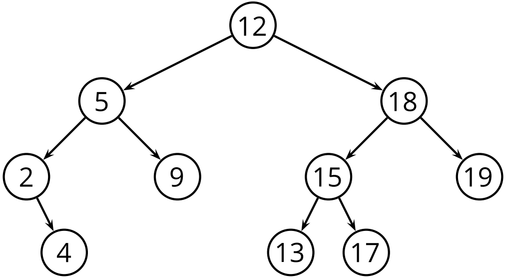

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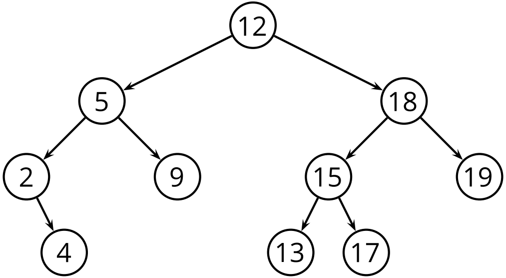

12 5 2 4 9 18 15 13 17 19

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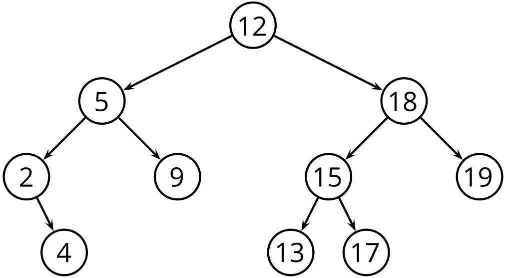

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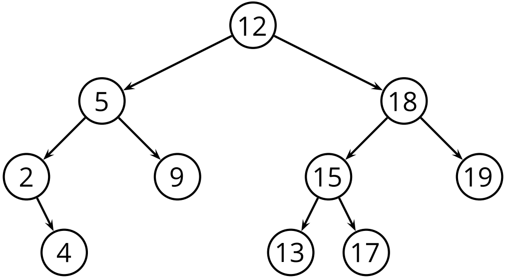

4 2 9 5 13 17 15 19 18 12

Reverse-Order Tree Walk

```
REVERSE-ORDER-TREE-WALK(x)
if x ≠ NIL
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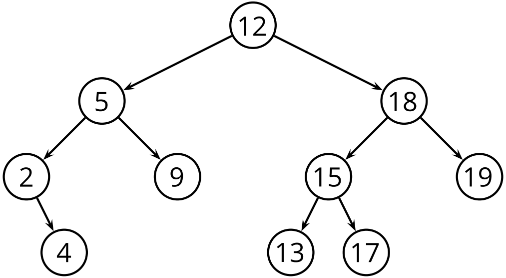

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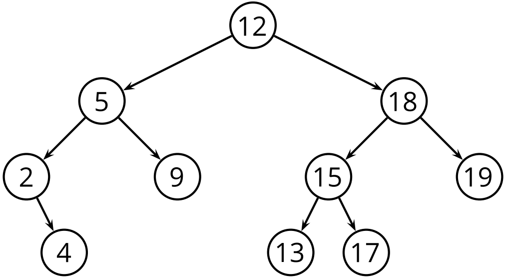

19 18 17 15 13 12 9 5 4 2

Application of postorder: Computing Arithmetic Expressions

- Arithmetic expressions can be represented by syntax trees.
- Given an expression represented by tree, compute its value !
- Each tree node: value field.
- Postorder traversal prints postfix notation/computes the value.

Complexity of Tree Walks

- The general recurrence is

$$T(n) = T(n_L) + T(n - n_L - 1) + \Theta(1)$$

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We could prove this using the *substitution method*

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We could prove this using the *substitution method*

- Can we do better? No!
  - the length of the output is $\Theta(n)$

Minimum and Maximum Keys

- Recall the *binary-search-tree property*
  - for all nodes x, y, and z
  - $y \in \textit{left-subtree}(x) \Rightarrow y.\textit{key} \leq x.\textit{key}$
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- So, the minimum key is in all the way to the left
  - similarly, the maximum key is all the way to the right

```
TREE-MINIMUM(x)
    while x.left ≠ NIL
        x = x.left
    return x
```

```
TREE-MAXIMUM(x)
    while x.right ≠ NIL
        x = x.right
    return x
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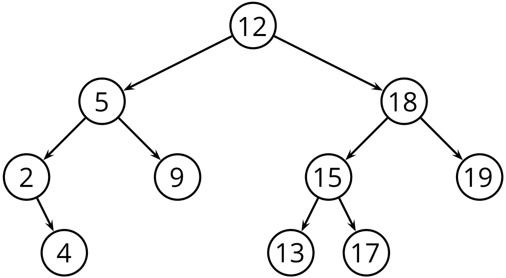

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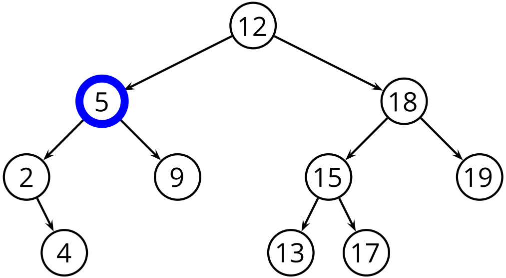

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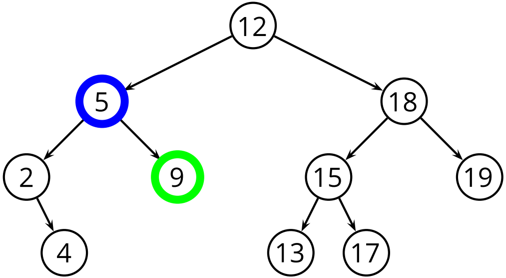

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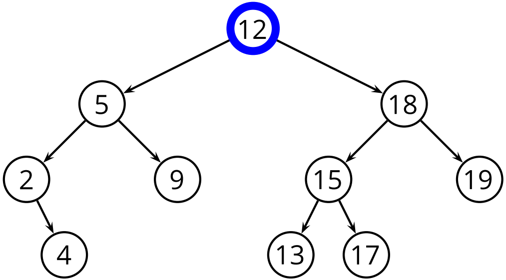

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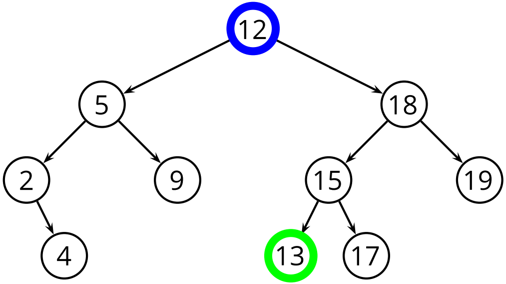

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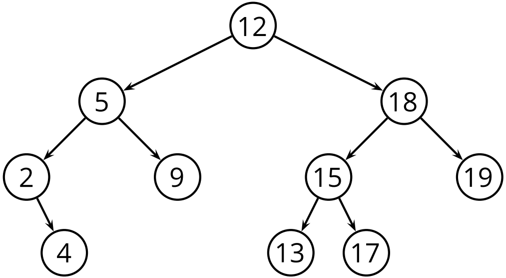

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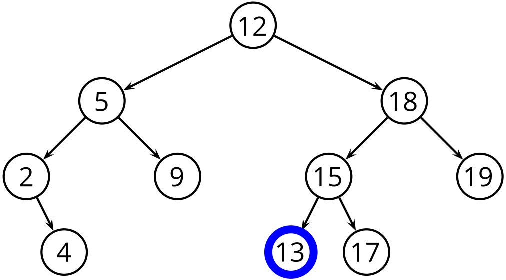

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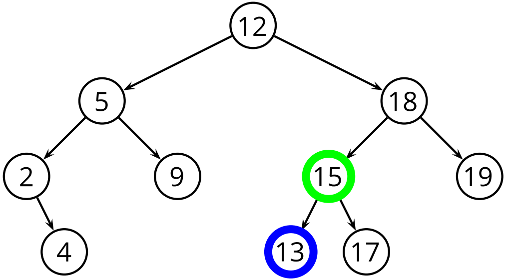

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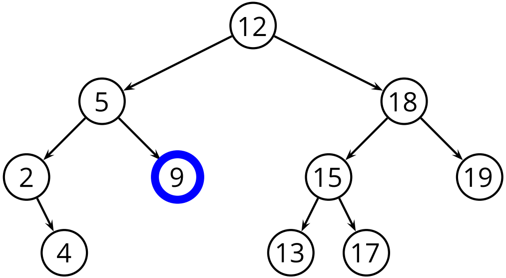

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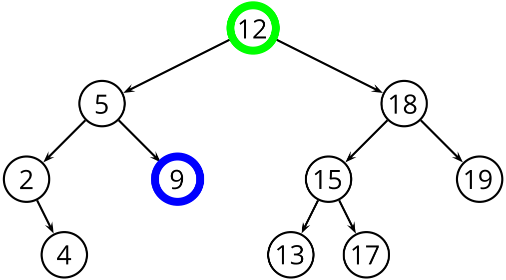

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Successor and Predecessor

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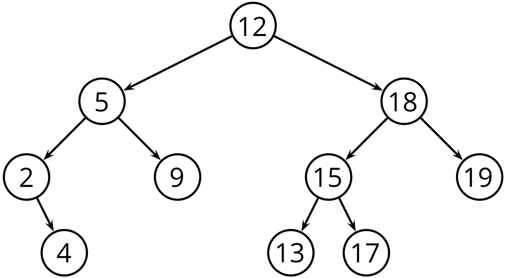

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- Otherwise it is the *first ancestor a* of *x* such that *x* falls in the *left* subtree of *a*

```
TREE-SUCCESSOR(x)
if x.right ≠ NIL
        return TREE-MINIMUM(x.right)
y = x.parent
while y ≠ NIL and x = y.right
        x = y
        y = y.parent
return y
```

Successor and Predecessor(2)

```
TREE-SUCCESSOR(x)
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y = x.parent
while y ≠ NIL and x = y.right
    x = y
    y = y.parent
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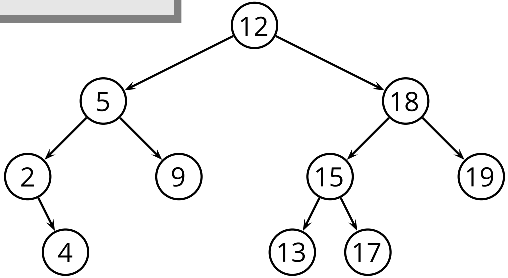

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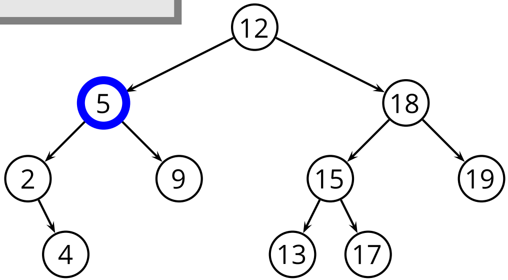

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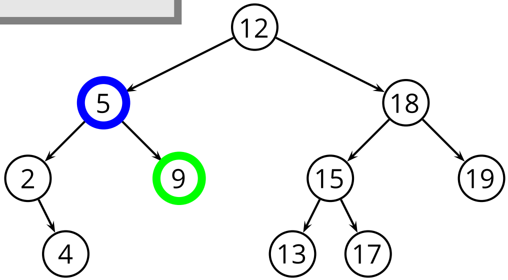

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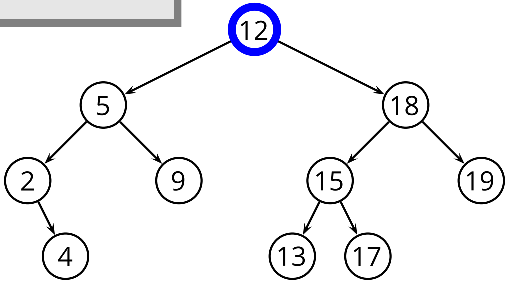

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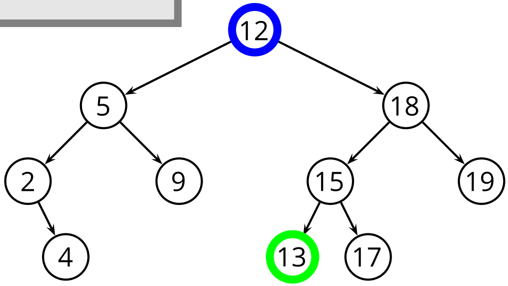

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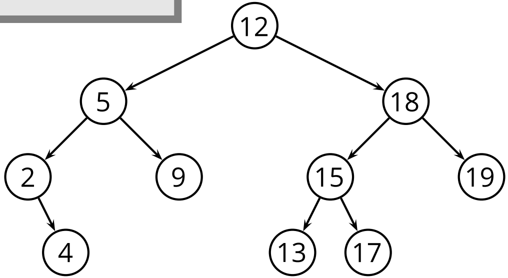

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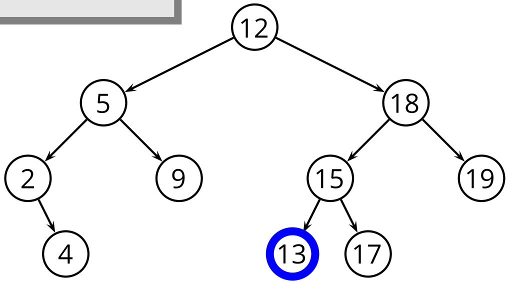

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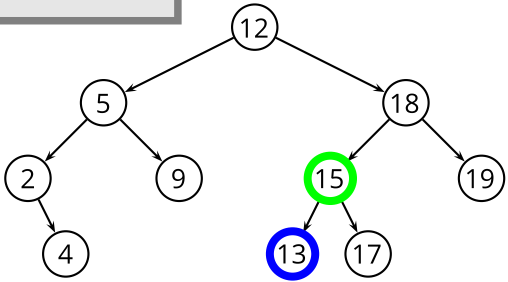

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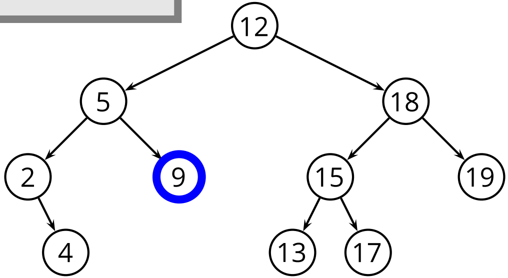

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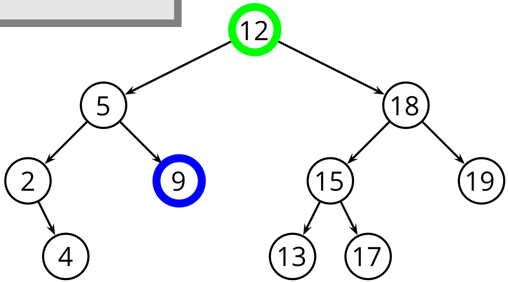

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TREE-SEARCH(x, k)
1  if x == NIL or k == x.key
2      return x
3  if k < x.key
4      return TREE-SEARCH(x.left, k)
5  else return TREE-SEARCH(x.right, k)
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Search

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- Complexity?

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$$T(n) = \Theta(\textit{depth of the tree})$$

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- Complexity?

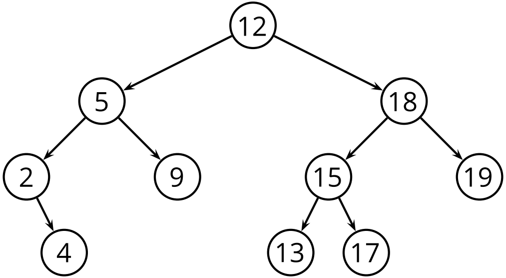
$$T(n) = \Theta(\textit{depth of the tree})$$

$$T(n) = O(n)$$

- Iterative *binary search*

- Iterative *binary search*

```
ITERATIVE-TREE-SEARCH(T, k)
x = T.root
while x ≠ NIL ∧ k ≠ x.key
    if k < x.key
        x = x.left
    else x = x.right
return x
```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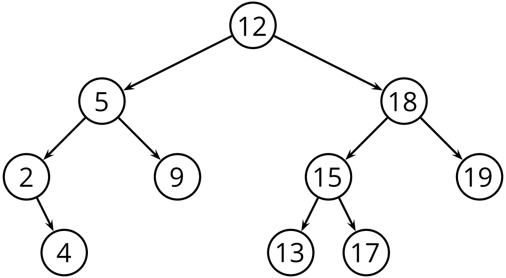

- *Idea*
  - in order to insert x, we *search* for x (more precisely $x.key$)
  - if we don't find it, we add it where the search stopped

Insertion (2)

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
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if y = NIL
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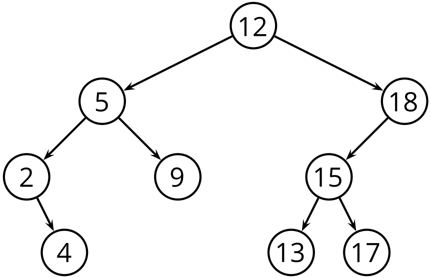

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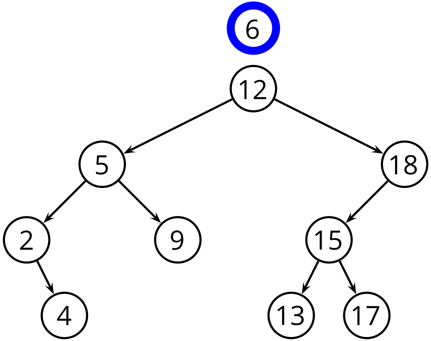

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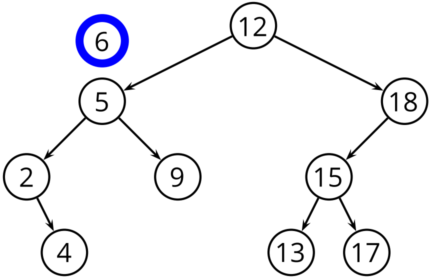

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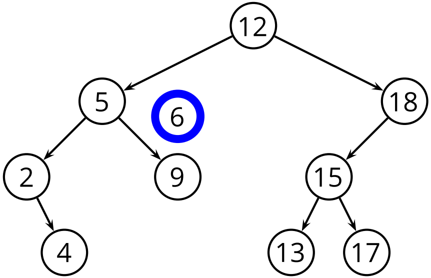

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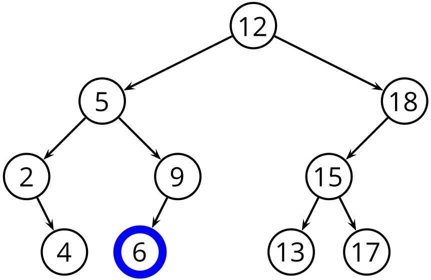

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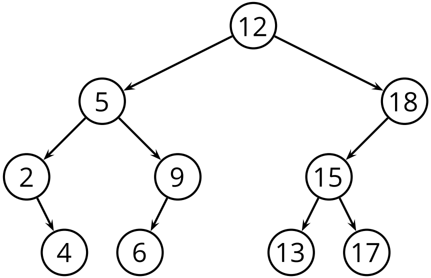

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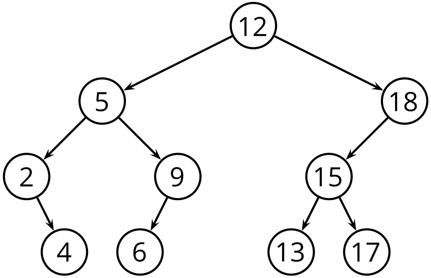

$$T(n) = \Theta(h)$$

Observation

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- *Idea:* use randomization to turn all cases in the average case

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- *Idea 2:* we can obtain a random permutation of the input sequence by randomly alternating two insertion procedures
  - *tail insertion:* this is what **TREE-INSERT** does
  - *head insertion:* for this we need a new procedure **TREE-ROOT-INSERT**
    - inserts n in T as if n was inserted as the first element

```
TREE-RANDOMIZED-INSERT1(T, z)
    r = uniformly random value from {1, ..., t.size + 1
    if r = 1
        TREE-ROOT-INSERT(T, z)
    else TREE-INSERT(T, z)
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  - no, clearly the last element can only go to the top or to the bottom

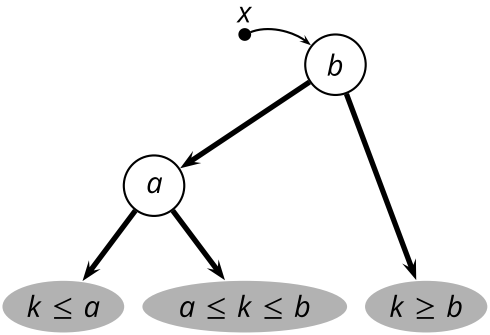
- It is true that any node has the same probability of being inserted at the top
  - this suggests a recursive application of this same procedure

```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
        return z
r = uniformly random value from {1, . . . , t.size + 1
if r = 1                                    // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

Randomized Insertion (3)

```
TREE-RANDOMIZED-INSERT(t, z)
if t = NIL
        return z
r = uniformly random value from {1, . . . , t.size + 1
if r = 1                                  // Pr[r = 1] = 1/(t.size + 1)
        z.size = t.size + 1
        return TREE-ROOT-INSERT(t, z)
if z.key < t.key
        t.left = TREE-RANDOMIZED-INSERT(t.left, z)
else t.right = TREE-RANDOMIZED-INSERT(t.right, z)
t.size = t.size + 1
return t
```

- Looks like this one really simulates a random permutation. . .


x
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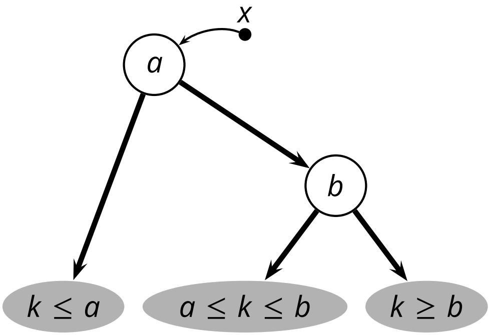
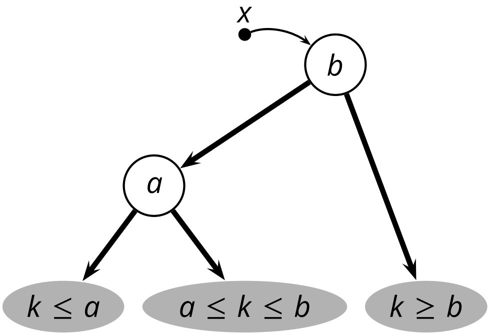
a
k ≤ a
a ≤ k ≤ b
k ≥ b



- $x = \textbf{Right-Rotate}(x)$



- x = **RIGHT-ROTATE**(x)
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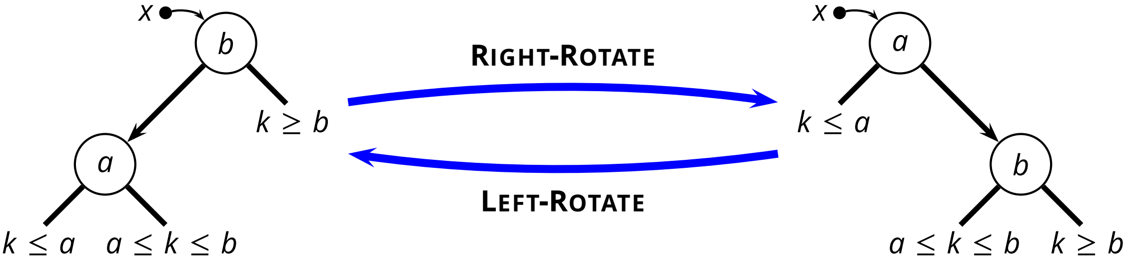
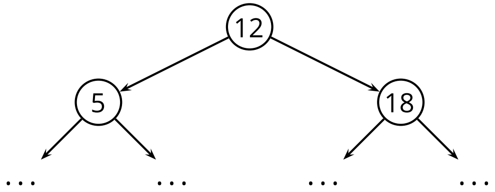


RIGHT-ROTATE(x)

```
l = x.left
x.left = l.right
l.right = x
return l
```

LEFT-ROTATE(x)

```
r = x.right
x.right = r.left
r.left = x
return r
```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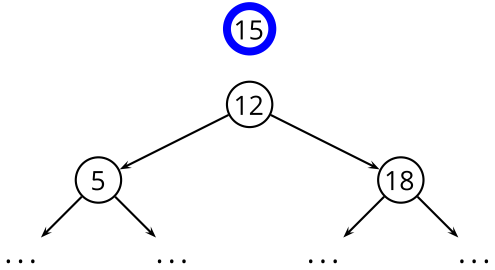
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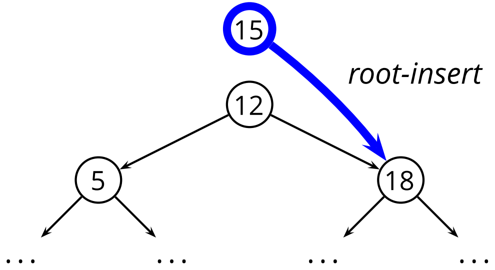
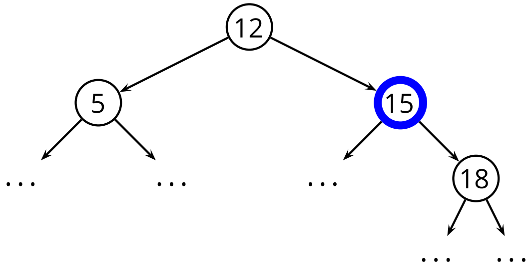


1. Recursively insert *z* at the root of the appropriate subtree (right)



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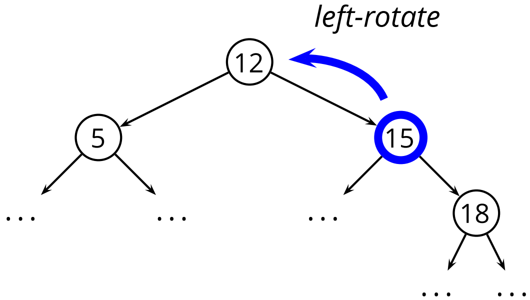

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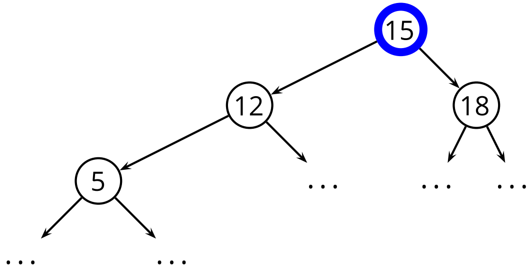
2. Rotate x with z (left-rotate)



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Root Insertion (2)

```
TREE-ROOT-INSERT(x, z)
if x = NIL
        return z
if z.key < x.key
        x.left = TREE-ROOT-INSERT(x.left, z)
        return RIGHT-ROTATE(x)
else x.right = TREE-ROOT-INSERT(x.right, z)
        return LEFT-ROTATE(x)
```

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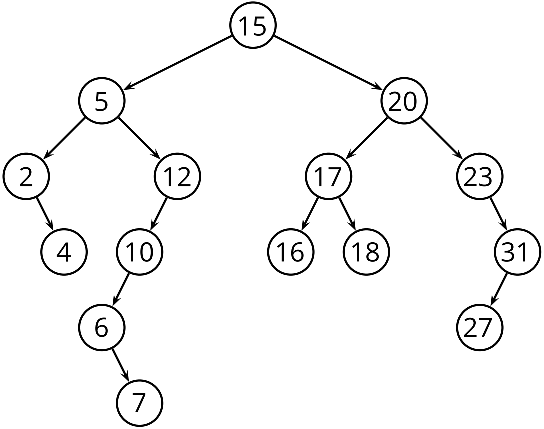
    - the worst case is still possible, but it is extremely improbable
  - *amortized maintenance:* e.g., balancing a BST or resizing a hash table
    - relatively expensive but “amortized” operations
  - *optimized data structures:* a self-balanced data structure
    - guaranteed $O(\log n)$ complexity bounds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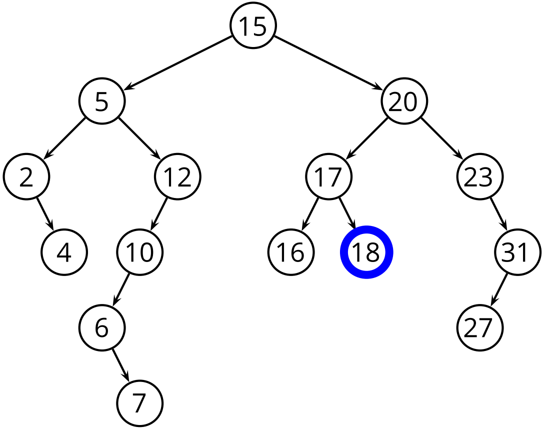
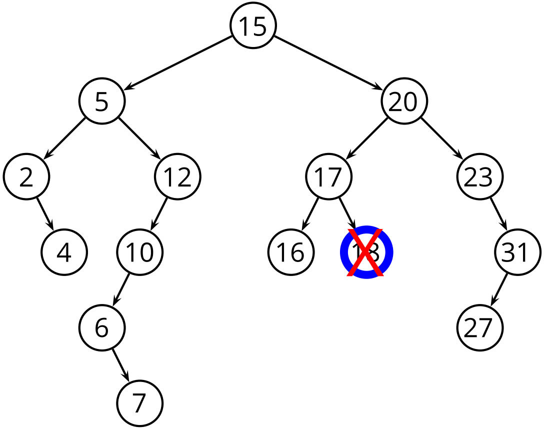
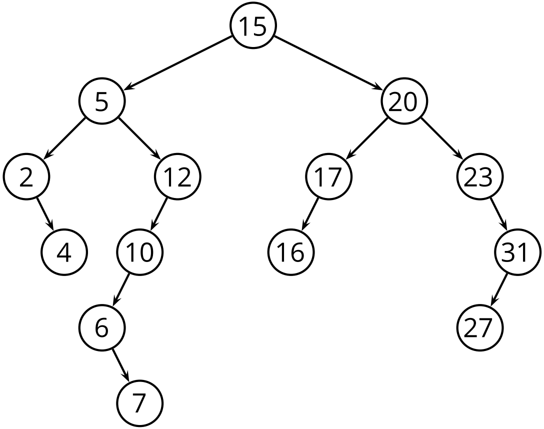


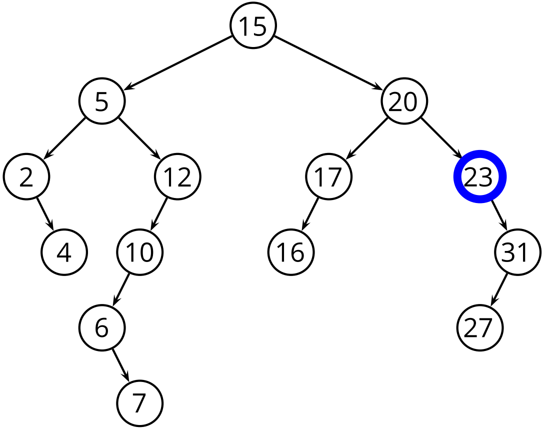
1. *z* has no children



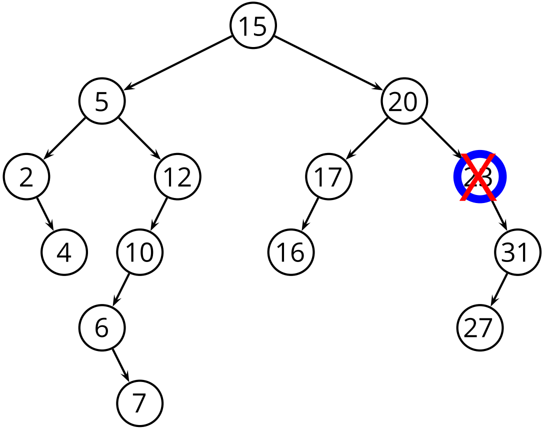
1. *z* has no children
   - simply remove *z*



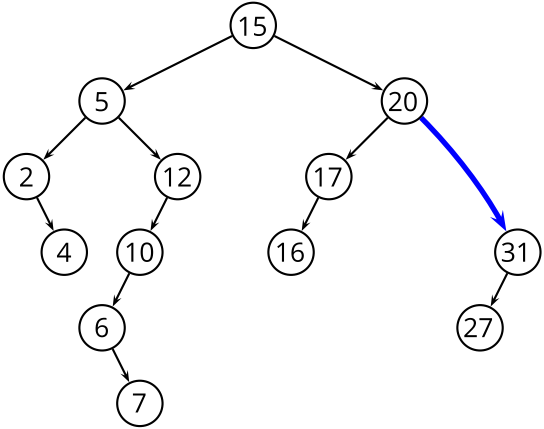
1. z has no children
   - simply remove z



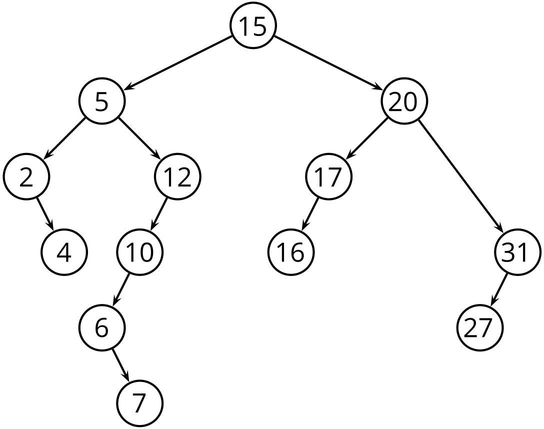
1. z has no children
   - simply remove z
2. z has one child



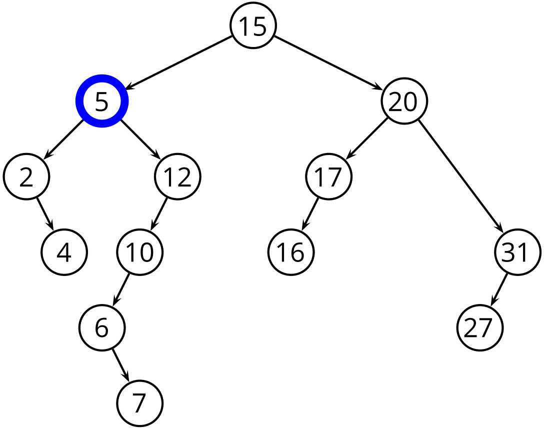
1. *z* has no children
   - simply remove *z*
2. *z* has one child
   - remove *z*



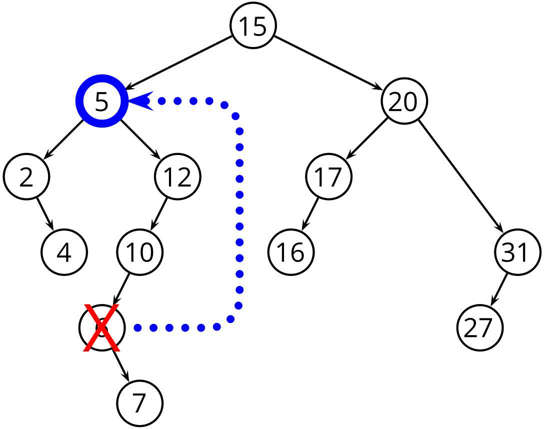
1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$



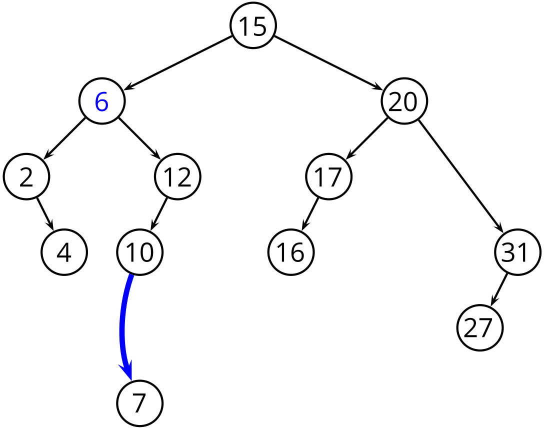
1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$



1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$
3. z has two children



1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$
3. z has two children
   - replace z with $y = \textbf{TREE-SUCCESSOR}(z)$
   - remove y (1 child!)



1. z has no children
   - simply remove z
2. z has one child
   - remove z
   - connect $z.parent$ to $z.right$
3. z has two children
   - replace z with $y =$ **Tree-Successor**(z)
   - remove y (1 child!)
   - connect $y.parent$ to $y.right$

```
TREE-DELETE(T, z)
if z.left = NIL or z.right = NIL
    y = z
else y = TREE-SUCCESSOR(z)
if y.left ≠ NIL
    x = y.left
else x = y.right
if x ≠ NIL
    x.parent = y.parent
if y.parent == NIL
    T.root = x
else if y = y.parent.left
        y.parent.left = x
    else y.parent.right = x
if y ≠ z
    z.key = y.key
    copy any other data from y into z
```