Algorithms and Data Structures (II)

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Where are we:

Graphs: Representations and Algorithms

- Graph traversals: BFS, DFS, Topological Sorting.
- Continue with Minimum Spanning Tree.

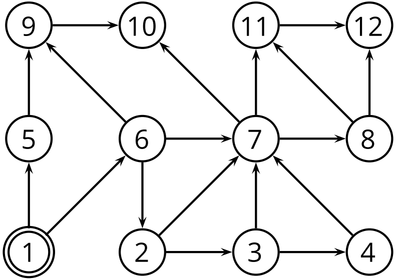
- One of the simplest but fundamental algorithms

- One of the simplest but fundamental algorithms

- *Input:* $G = (V, E)$ and a vertex $s \in V$
  - explores the graph, touching all vertices that are reachable from s
  - iterates through the vertices at increasing distance (edge distance)
  - computes the distance of each vertex from s
  - produces a ***breadth-first tree*** rooted at s
  - works on both *directed* and *undirected* graphs

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
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color[s] = GRAY
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Q = ∅
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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
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            d[v] = d[u] + 1
            π[v] = u
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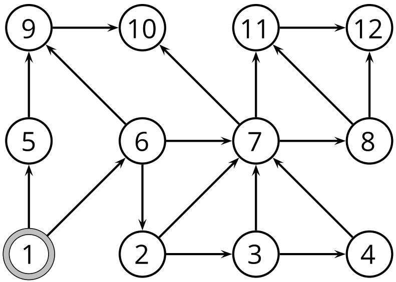
```



$u = 1$

$Q = \emptyset$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
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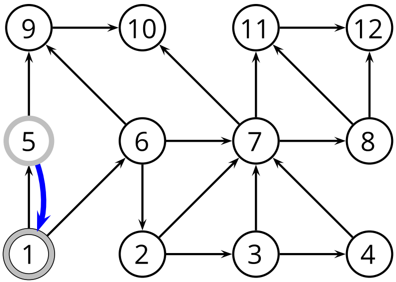
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

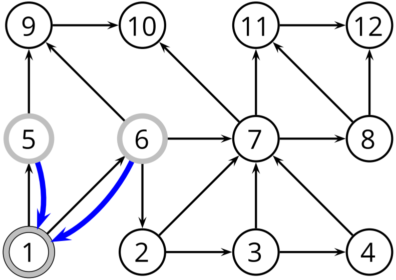


$u = 1$

$Q = \{5\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

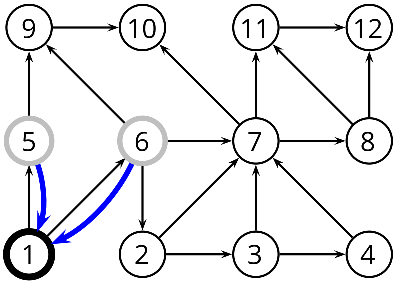


$u = 1$

$Q = \{5, 6\}$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

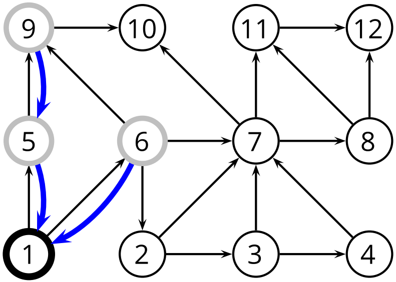


$u = 5$

$Q = \{6\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

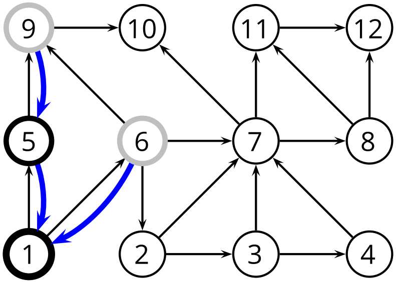


$u = 5$

$Q = \{6, 9\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

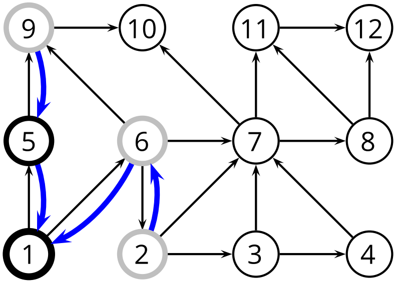


$u = 6$

$Q = \{9\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

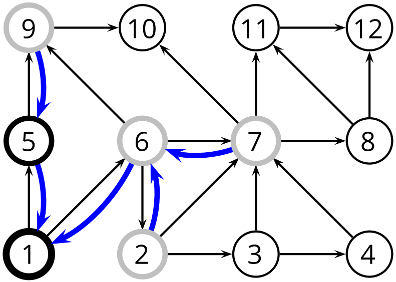


$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

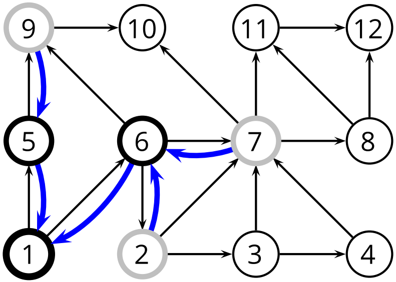


$u = 6$

$Q = \{9, 2, 7\}$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

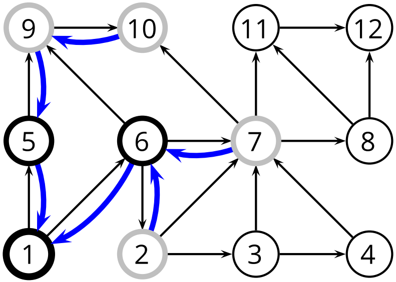


$u = 9$

$Q = \{2, 7\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

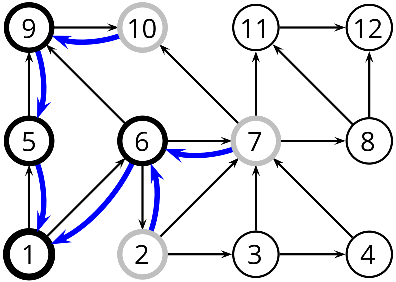


$u = 9$

$Q = \{2, 7, 10\}$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

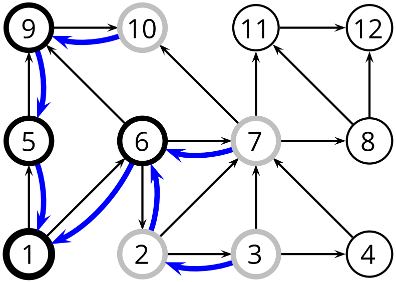


$u = 2$

$Q = \{7, 10\}$

BFS Algorithm

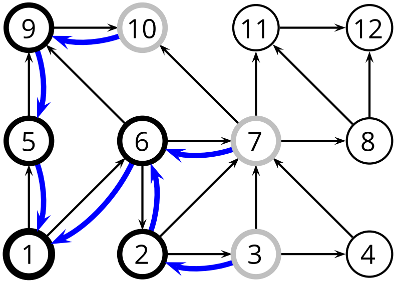
```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```



$u = 2$

$Q = \{7, 10, 3\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```

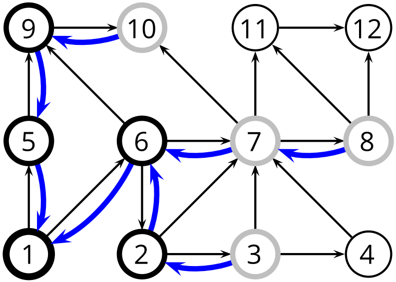


$u = 7$

$Q = \{10, 3\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

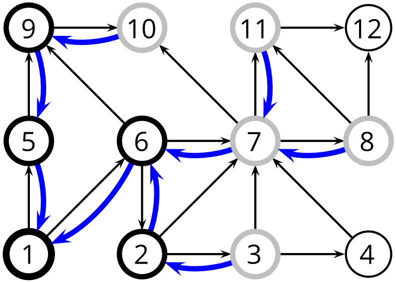


$u = 7$

$Q = \{10, 3, 8\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

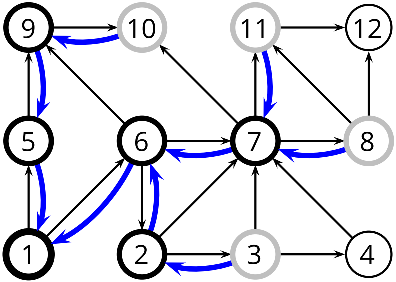


$u = 7$

$Q = \{10, 3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

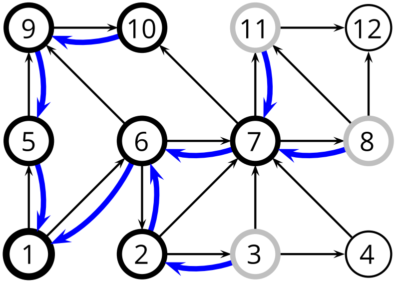


$u = 10$

$Q = \{3, 8, 11\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

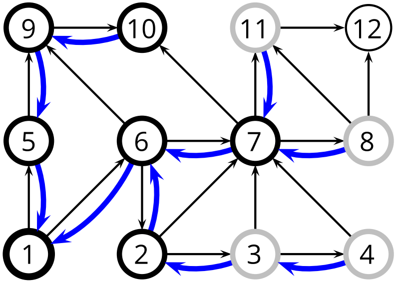


$u = 3$

$Q = \{8, 11\}$

BFS Algorithm

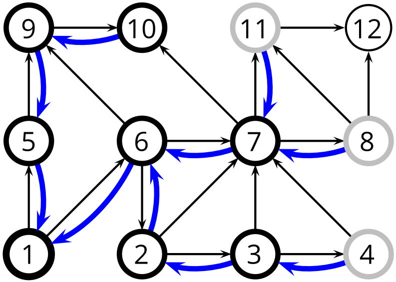
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
                if color[v] == WHITE
                        color[v] = GRAY
                        d[v] = d[u] + 1
                        π[v] = u
                        ENQUEUE(Q, v)
        color[u] = BLACK
```



$u = 3$

$Q = \{8, 11, 4\}$

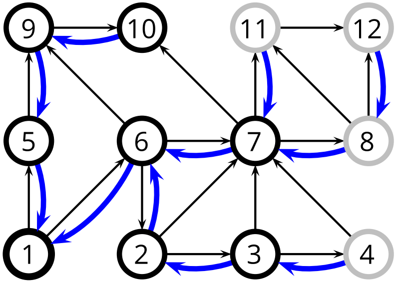
```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```



$u = 8$

$Q = \{11, 4\}$

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

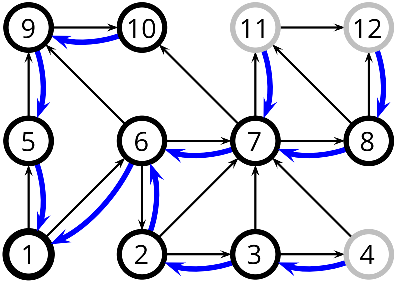


$u = 8$

$Q = \{11, 4, 12\}$

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ V(G) \ {s}
    color[u] = WHITE
    d[u] = ∞
    π[u] = NIL
color[s] = GRAY
d[s] = 0
π[s] = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ Adj[u]
        if color[v] == WHITE
            color[v] = GRAY
            d[v] = d[u] + 1
            π[v] = u
            ENQUEUE(Q, v)
    color[u] = BLACK
```

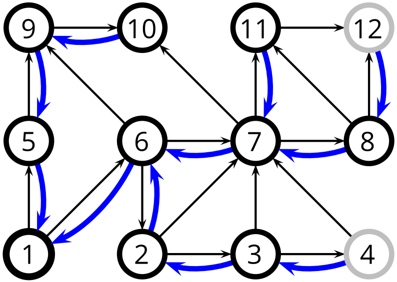


$u = 11$

$Q = \{4, 12\}$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
            if color[v] == WHITE
                color[v] = GRAY
                d[v] = d[u] + 1
                π[v] = u
                ENQUEUE(Q, v)
        color[u] = BLACK
```

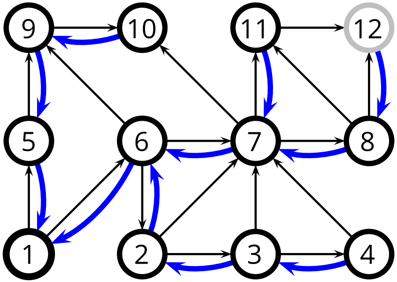


$u = 4$

$Q = \{12\}$

BFS Algorithm

```
BFS(G, s)
  for each vertex u ∈ V(G) \ {s}
      color[u] = WHITE
      d[u] = ∞
      π[u] = NIL
  color[s] = GRAY
  d[s] = 0
  π[s] = NIL
  Q = ∅
  ENQUEUE(Q, s)
  while Q ≠ ∅
      u = DEQUEUE(Q)
      for each v ∈ Adj[u]
          if color[v] == WHITE
              color[v] = GRAY
              d[v] = d[u] + 1
              π[v] = u
              ENQUEUE(Q, v)
      color[u] = BLACK
```



$u = 12$

$Q = \emptyset$

BFS Algorithm

```
BFS(G, s)
    for each vertex u ∈ V(G) \ {s}
        color[u] = WHITE
        d[u] = ∞
        π[u] = NIL
    color[s] = GRAY
    d[s] = 0
    π[s] = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ Adj[u]
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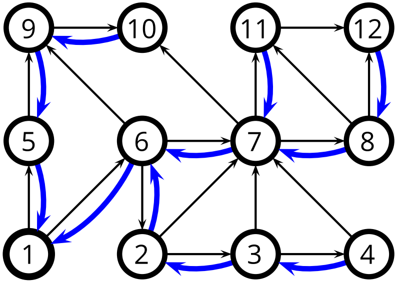

Complexity of BFS

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BFS(G, s)
for each vertex u ∈ V(G) \ {s}
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color[s] = GRAY
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Q = ∅
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Depth-First Search

- Immediately follow the links of the most recently-visited vertex, then backtrack when you reach a dead-end
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  - explores the graph, touching *all vertices*
  - produces a ***depth-first forest***, consisting of all the ***depth-first trees*** defined by the DFS exploration
  - associates ***two time-stamps*** to each vertex
    - $d[u]$ records when u is first discovered
    - $f[u]$ records when DFS finishes examining u's edges, and therefore backtracks from u

DFS Algorithm

```
DFS(G)
    for each vertex u ∈ V(G)
        color[u] = WHITE
        π[u] = NIL
    time = 0 // "global" variable
    for each vertex u ∈ V(G)
        if color[u] == WHITE
            DFS-VISIT(u)
```

```
DFS-VISIT(u)
    color[u] = GREY
    time = time + 1
    d[u] = time
    for each v ∈ Adj[u]
        if color[v] == WHITE
            π[v] = u
            DFS-VISIT(v)
    color[u] = BLACK
    time = time + 1
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- So, the overall complexity is $\Theta(|V| + |E|)$

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  Given a *directed acyclic graph* (DAG)

  - find an ordering of vertices such that you only end up with *forward links*

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- **Problem:** (topological sort)

  Given a *directed acyclic graph* (DAG)

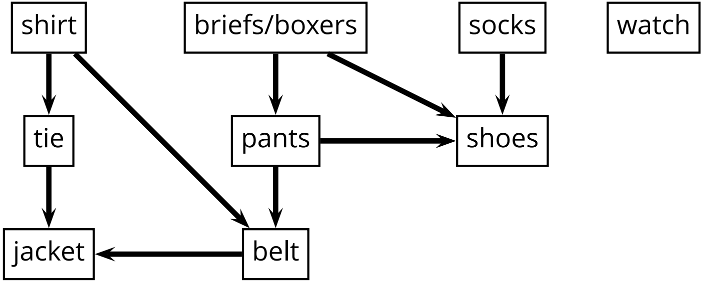
  - find an ordering of vertices such that you only end up with *forward links*

- **Example:** dependencies in software packages

  - find an installation order for a set of software packages
  - such that every package is installed only after all the packages it depends on

Topological Sort Algorithm

Topological Sort Algorithm



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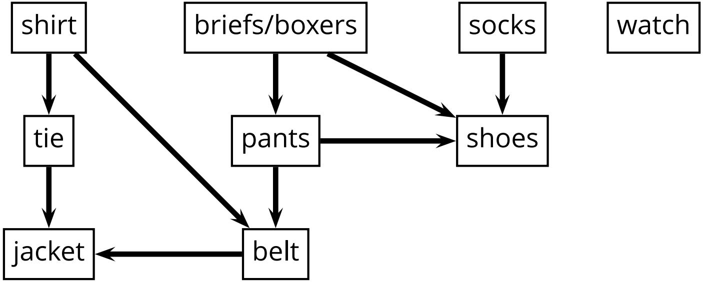

TOPOLOGICAL-SORT(G)
1 **DFS**(G)
2 output V sorted in reverse order of $f[\cdot]$

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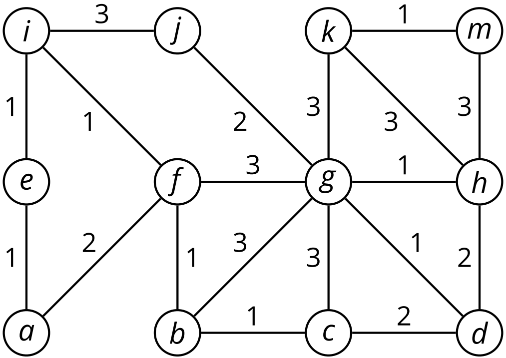
  - a ***spanning tree***
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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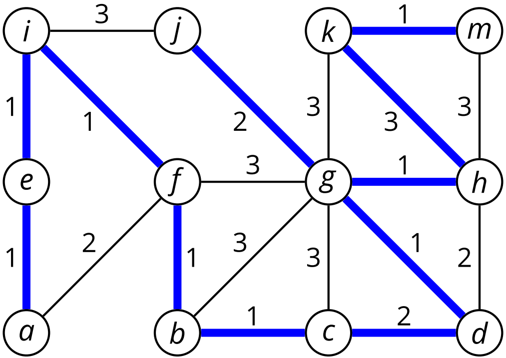

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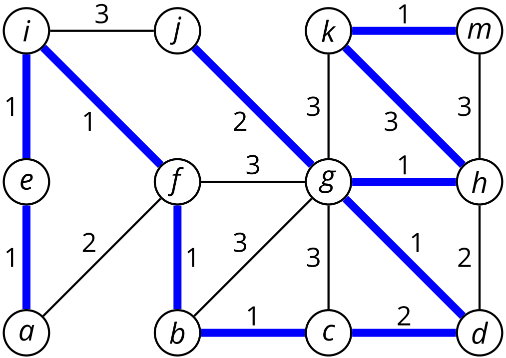

- There may be more than one minimum spanning tree

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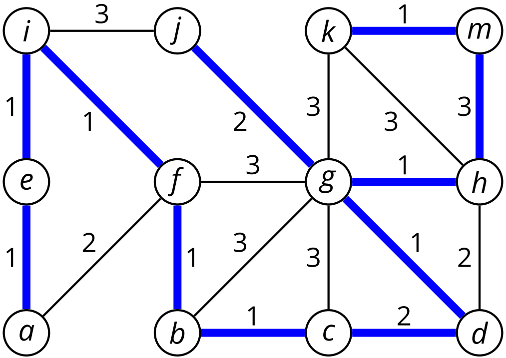

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- Does it work?

Refined Idea

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    A = ∅
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  - more or less the *definition* of a greedy algorithm

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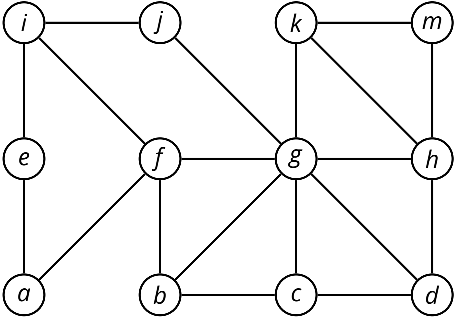

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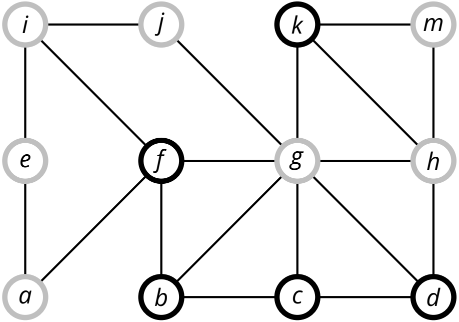

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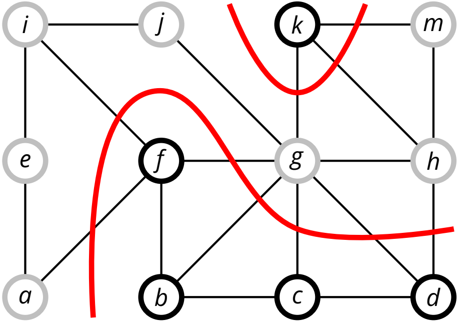

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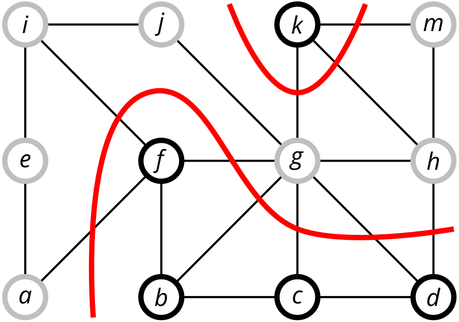

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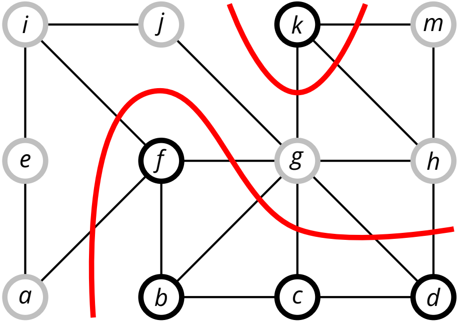

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

Finding a Safe Edge

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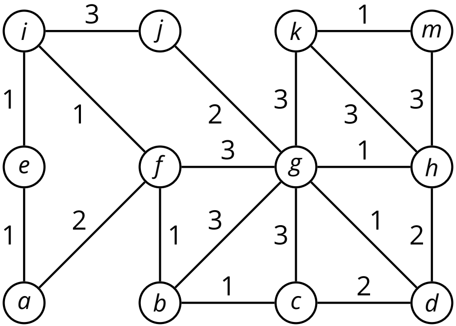

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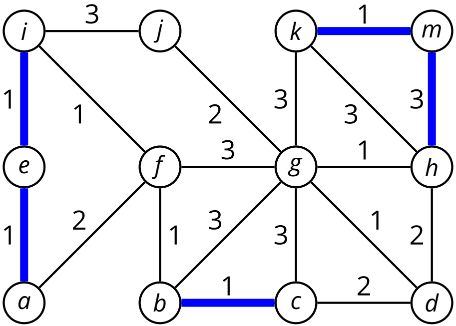

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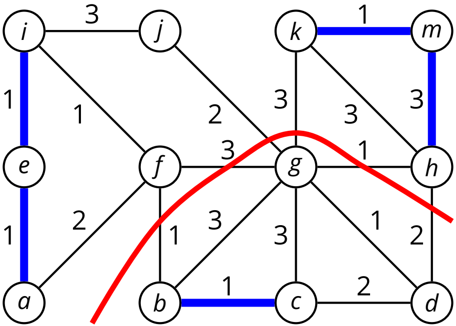

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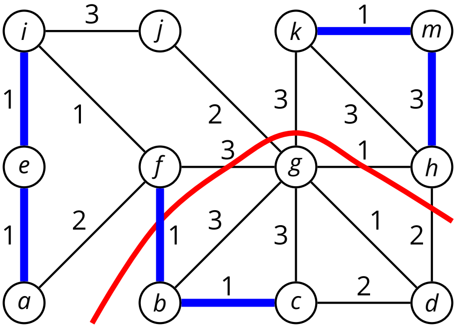

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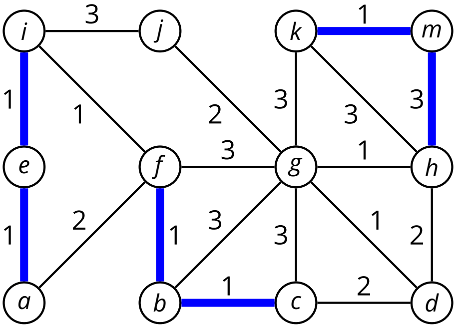

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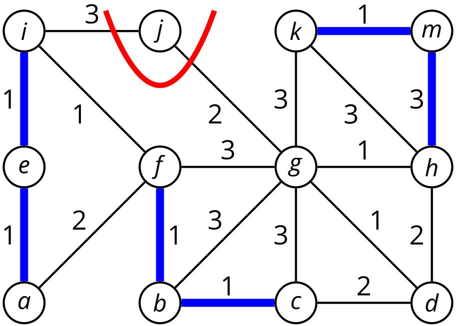

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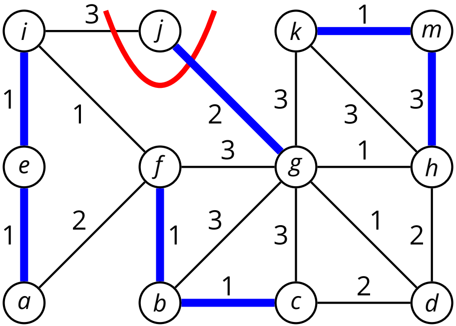

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Concrete Algorithms

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- Prim's algorithm (1957)
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Disjoint-Set Data Structure

- *Make-Set*(x) creates a set containing x
- *Find-Set*(x) returns the *representative* of the set containing x
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- *Union*(x, y) joins the sets containing x and y

Kruskal's Algorithm

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MST-KRUSKAL(G, w)
  A = ∅
  for each vertex v ∈ V(G)
      MAKE-SET(v)                    // disjoint-set data structure
  sort E in non-decreasing order by weight w
  for each edge (u, v) ∈ E, taken in non-decreasing order by w
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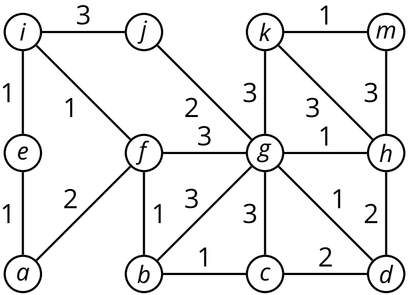

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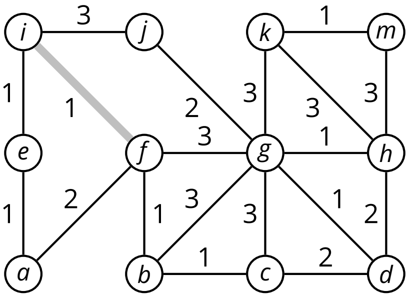

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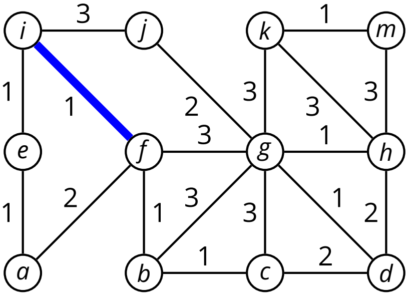

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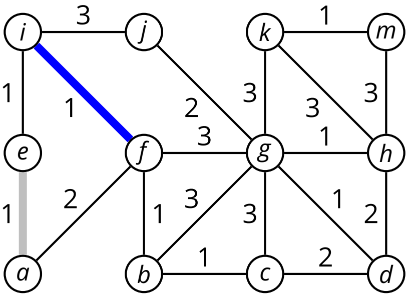

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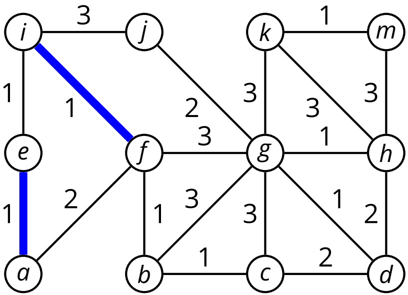

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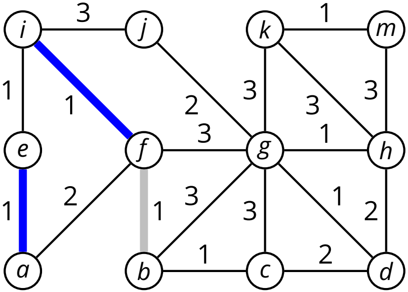

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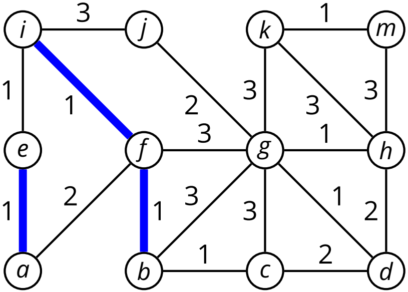

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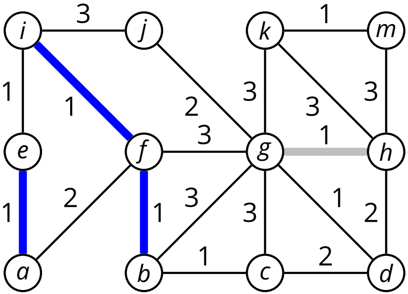

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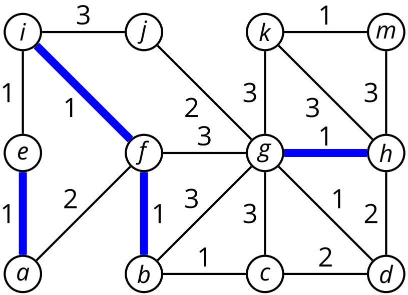

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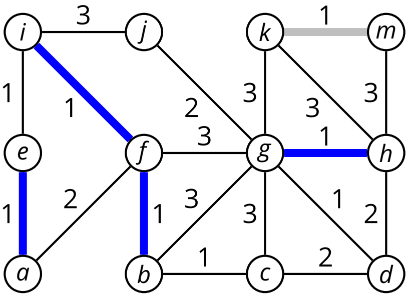

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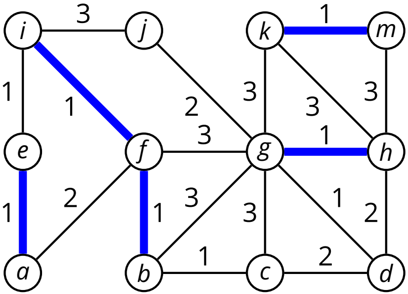

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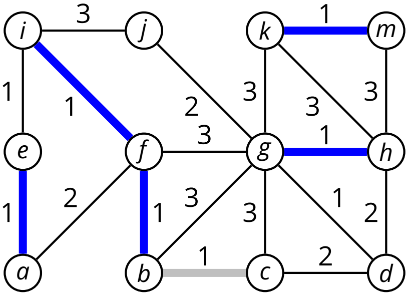

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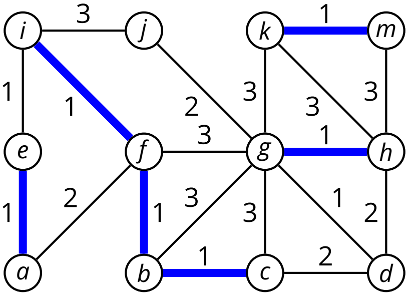

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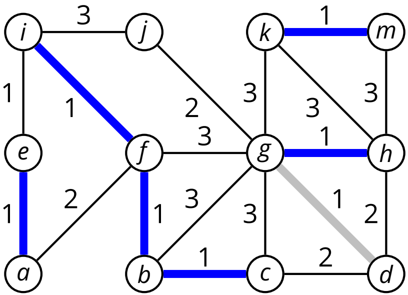

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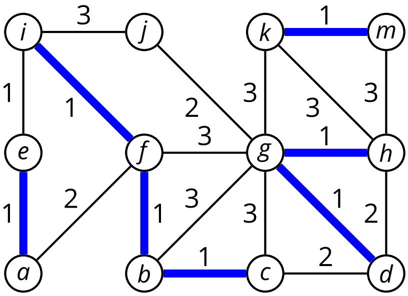

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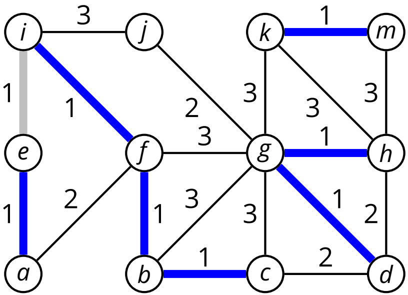

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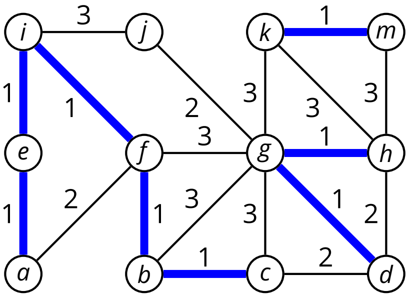

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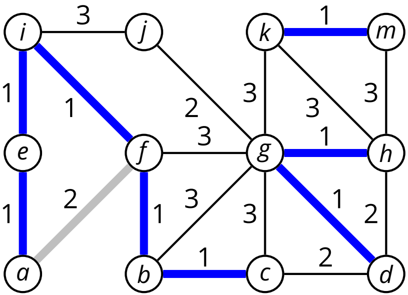

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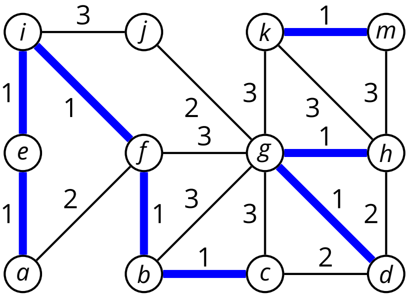

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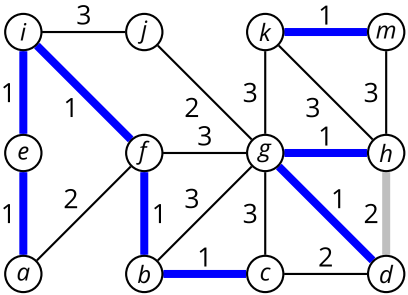

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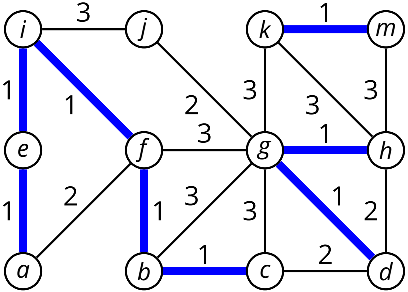

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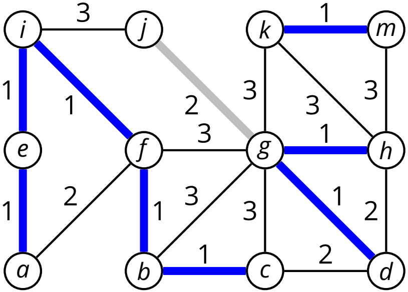

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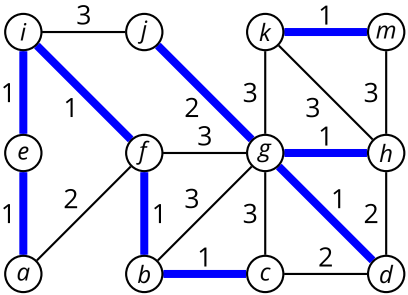

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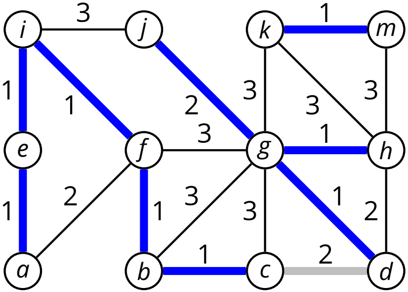

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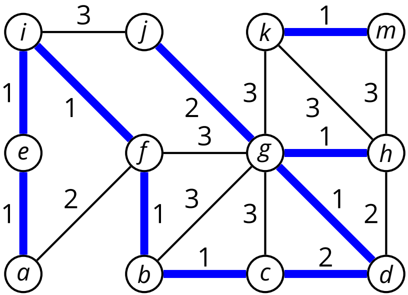

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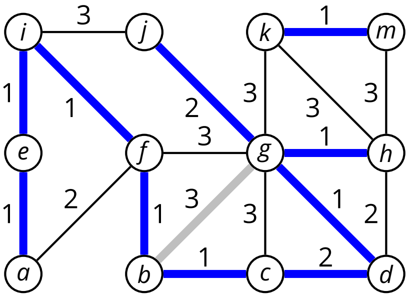

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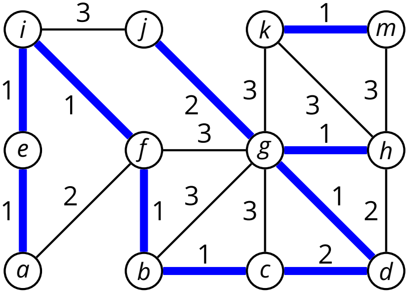

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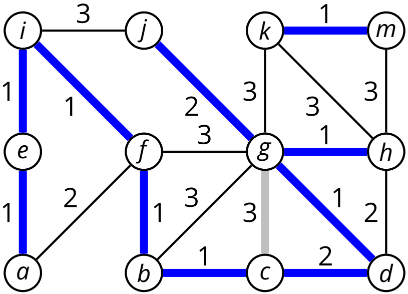

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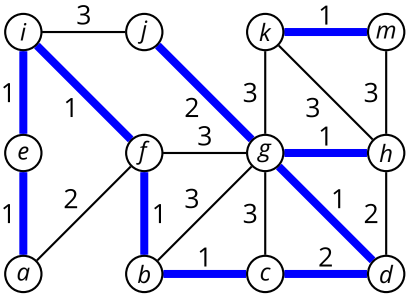

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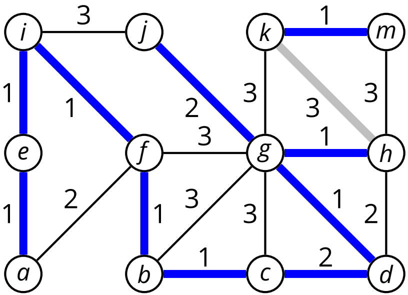

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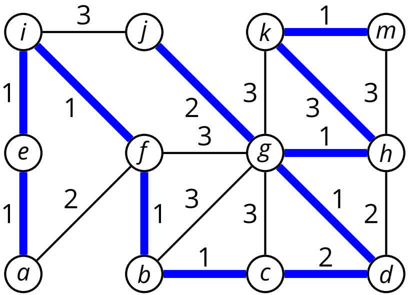

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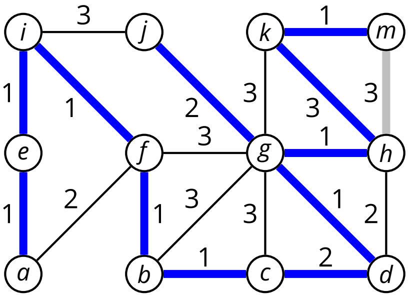

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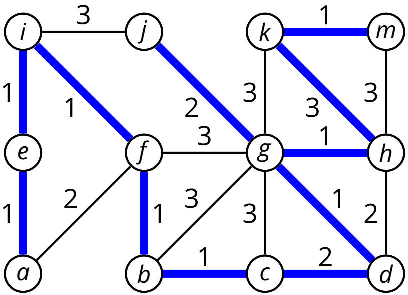

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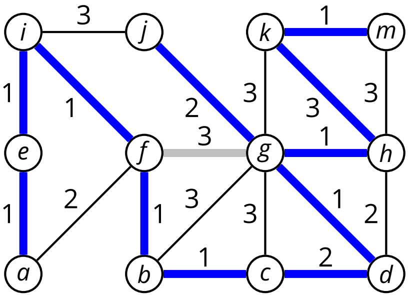

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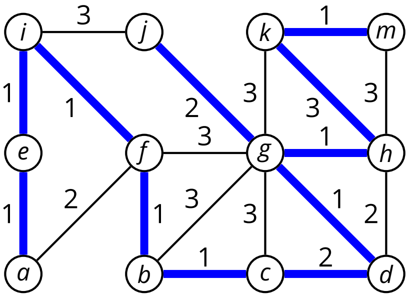

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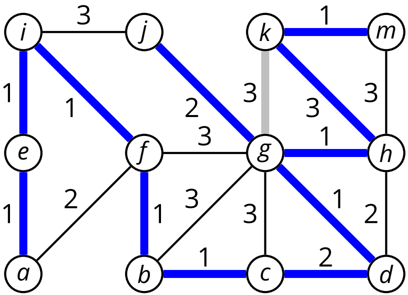
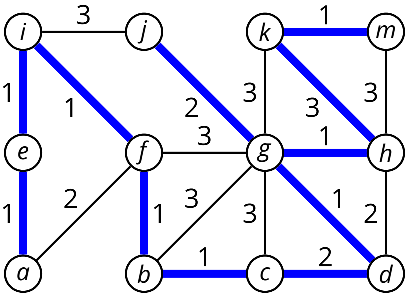


Kruskal's Algorithm

```
MST-KRUSKAL(G, w)
  A = ∅
  for each vertex v ∈ V(G)
      MAKE-SET(v)                    // disjoint-set data structure
  sort E in non-decreasing order by weight w
  for each edge (u, v) ∈ E, taken in non-decreasing order by w
      if FIND-SET(u) ≠ FIND-SET(v)
          A = A ∪ {(u, v)}
          UNION(u, v)
```



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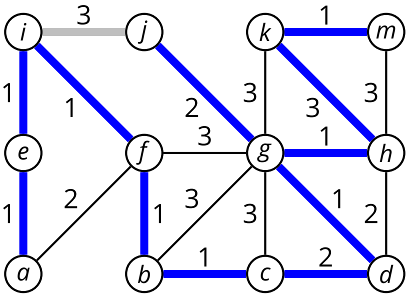

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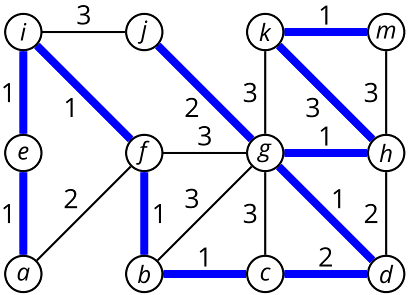

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Complexity of MST-KRUSKAL

```
MST-KRUSKAL(G, w)
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5  for each edge (u, v) ∈ E, taken in non-decreasing order by w
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```

- $|V|$ times **MAKE-SET** (loop of line 2–3)

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- $|V|$ times **MAKE-SET** (loop of line 2–3)
- $O(|E| \log |E|)$ for sorting E (line 4)
- $2|E|$ times **FIND-SET**

Complexity of MST-KRUSKAL

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- $O(|E| \log |E|)$ for sorting E (line 4)
- $2|E|$ times **FIND-SET**
- $O(|E|)$ times **UNION**

Prim's Algorithm

```
MST-PRIM(G, w, r)
for each vertex u ∈ V(G)
    key[u] = ∞
    π(u) = NIL
key[r] = 0
Q = V(G)
while Q ≠ ∅
    u = EXTRACT-MIN(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
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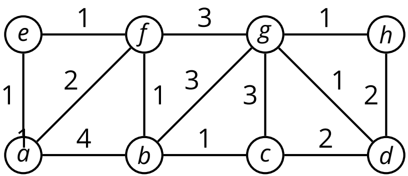

$$Q = \{(a, 0, \cdot), (b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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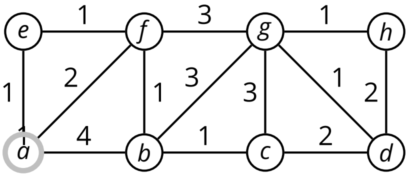

$$Q = \{(b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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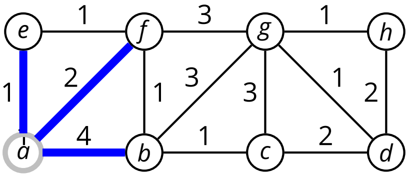

$Q = \{(e, 1, a), (f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

```
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Q = V(G)
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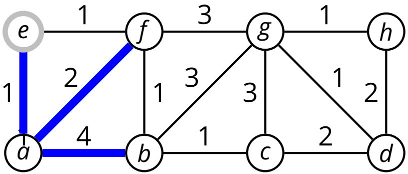
```



$Q = \{(f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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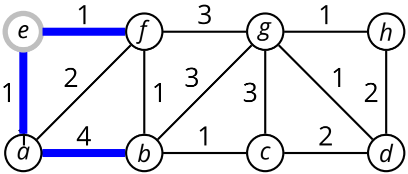
```



$Q = \{(f, 1, e), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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MST-PRIM(G, w, r)
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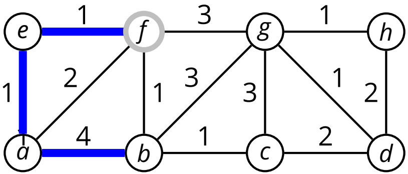

$Q = \{(b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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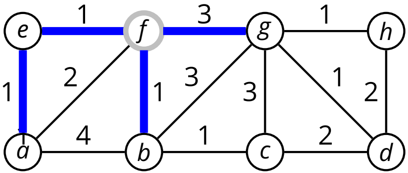

$$Q = \{(b, 1, f), (g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
MST-PRIM(G, w, r)
for each vertex u ∈ V(G)
        key[u] = ∞
        π(u) = NIL
key[r] = 0
Q = V(G)
while Q ≠ ∅
        u = EXTRACT-MIN(Q) // min by key[u]
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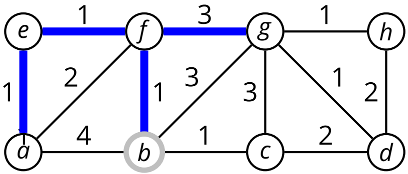

$Q = \{(g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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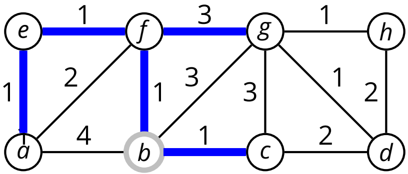

$$Q = \{(c, 1, b), (g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

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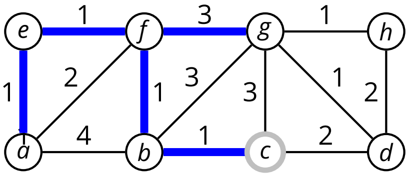

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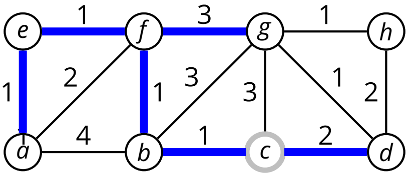

$Q = \{(d, 2, c), (g, 3, f), (h, \infty, \cdot)\}$

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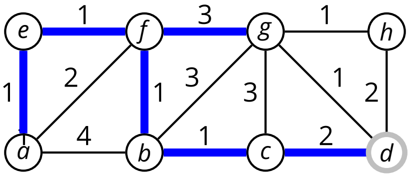

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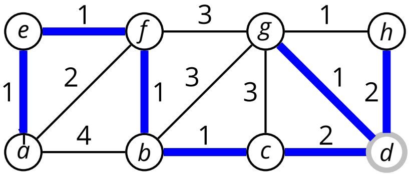

$Q = \{(g, 1, d), (h, 2, d)\}$

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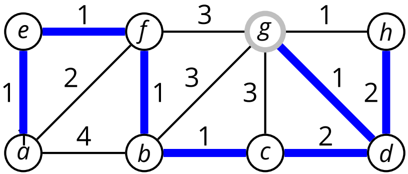

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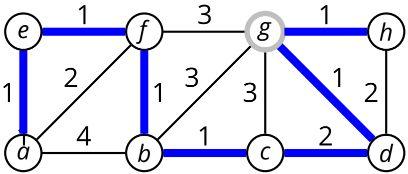

$Q = \{(h, 1, g)\}$

Prim's Algorithm

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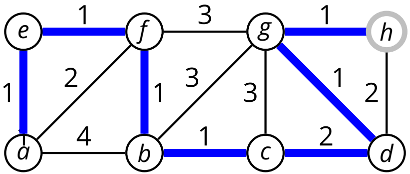

$Q = \{\}$

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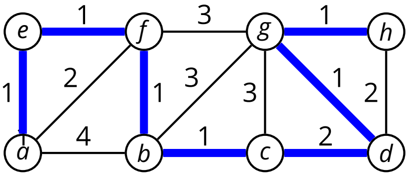

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